Navier-Stokes, Euler and Other Related Equations

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Rayleigh Bénard Convection / Boussinesq Approximation

- Conservation of Momentum

$$\frac{\partial}{\partial t}\vec{u} - \upsilon\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho_0}\nabla p + f\,\vec{k}\times\vec{u} = gT\vec{k}$$

- Incompressibility

$$\nabla\cdot\vec{u} = 0$$

- Heat Transport and Diffusion

$$\frac{\partial}{\partial t}T - \kappa\Delta T + (\vec{u}\cdot\nabla)T = 0$$

The Boundary Conditions

We partition the boundary of into:

$$\Gamma_u = \{(x, y, z) \in \overline{\quad} : z = 0\},$$
$$\Gamma_b = \{(x, y, z) \in \overline{\quad} : z = -h\},$$
$$\Gamma_s = \{(x, y, z) \in \overline{\quad} : (x, y) \in \partial M, \ -h \leq z \leq 0\}.$$

$$\text{on } \Gamma_u : \frac{\partial v}{\partial z} = h\,\tau, \ w = 0, \ \frac{\partial T}{\partial z} = -\alpha(T - T^*);$$
$$\text{on } \Gamma_b : \frac{\partial v}{\partial z} = 0, \ w = 0, \ \frac{\partial T}{\partial z} = 0;$$
$$\text{on } \Gamma_s : v \cdot \vec{n} = 0, \ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0, \ \frac{\partial T}{\partial \vec{n}} = 0,$$

Sobolev Spaces

$$H^s(\Omega) = \{\varphi = \sum_{\vec{k} \in \mathrm{Z}^d} \hat{\varphi}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

such that

$$\sum_{\vec{k} \in \mathrm{Z}^{\mathrm{d}}} \left|\hat{\varphi}_{\vec{k}}\right|^2 (1 + \left|\vec{k}\right|^2)^s < \infty\}$$

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Temperature Estimates

- Maximum Principle

$$\|T\|_{L^\infty} \le C_0 + C_1 \|T_0\|_{L^\infty}$$

- Gradient Estimates

$$\frac{1}{2}\frac{d}{dt}\|\nabla T\|_{L^2}^2 + \kappa \|\Delta T\|_{L^2}^2 = \int (\vec{u}\cdot\nabla)T \cdot \Delta T dx$$

- Estimate of the Nonlinear Term

$$\left|\int (\vec{u}\cdot\nabla)T\cdot\Delta T dx\right| \le c\|\vec{u}\|_{L^6}\|\nabla T\|_{L^3}\|\Delta T\|_{L^2}$$

- Interpolation/Calculus Inequality

$$\|\varphi\|_{L^3} \le c\|\varphi\|_{L^2}^{1/2}\|\nabla\varphi\|_{L^2}^{1/2} \qquad \forall\varphi\in\dot{H}^1$$

$$\Rightarrow$$

$$\left|\int (\vec{u}\cdot\nabla)T\cdot\Delta T dx\right| \le c\|\vec{u}\|_{L^6}\|\nabla T\|_{L^2}^{1/2}\|\Delta T\|_{L^2}^{3/2}$$

- Young's Inequality

$$|a\cdot b| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \qquad \frac{1}{p}+\frac{1}{q}=1$$

$$\left|\int(\vec{u}\cdot\nabla)T\cdot\Delta T dx\right| \le \frac{c}{\kappa^3}\|\vec{u}\|_{L^6}^4\|\nabla T\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta T\|_{L^2}^2$$

$$\Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla T\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta T\|_{L^2}^2 \le \frac{c}{\kappa^3}\|u\|_{L^6}^4\|\nabla T\|_{L^2}^2$$

By Gronwall's inequality

$$\|\nabla T(t)\|_{L^2}^2 \le \|\nabla T(0)\|_{L^2}^2 e^{\frac{c}{\kappa^3}\int_0^t \|u(\tau)\|_{L^6}^4 d\tau}$$

Question:

Is

$$\int_0^t \|u(\tau)\|_{L^6}^4 \, d\tau \le K ?$$

To answer this question we have to deal with the Navier-Stokes equations.

The Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

Plus Boundary conditions, say periodic in the box

$$\Omega = [0, L]^3$$

- We will assume that $\rho_0 = 1$
- Denote by $\langle \varphi \rangle = \int_\Omega \varphi(x)dx$
- Observe that if $\langle \vec{u}_0 \rangle = \langle \vec{f} \rangle = 0$ then $\langle \vec{u} \rangle = 0.$
- Poincaré Inequality

For every $\varphi \in H^1$ with $\langle \varphi \rangle = 0$ we have

$$\|\varphi\|_{L^2} \leq cL \|\nabla \varphi\|_{L^2}$$

Navier-Stokes Equations Estimates

- Formal Energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 + \int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{u} + \int\nabla p\cdot\vec{u} = \left(\vec{f},\vec{u}\right)$$

- Observe that since $\nabla\cdot\vec{u} = 0$ we have

$$\int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{u}dx = \int\nabla p\cdot\vec{u}dx = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 = \left(\vec{f},\vec{u}\right)$$

By the Cauchy-Schwarz and Poincaré inequalities

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{u}\right\|_{L^2}^2 + \upsilon\left\|\nabla\vec{u}\right\|_{L^2}^2 \leq \left\|\vec{f}\right\|_{L^2}^2 \left\|\vec{u}\right\|_{L^2}^2 \leq cL\left\|\vec{f}\right\|_{L^2} \left\|\nabla\vec{u}\right\|_{L^2}$$

By the Young's inequality

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{u}\right\|_{L^2}^2 + \upsilon\left\|\nabla\vec{u}\right\|_{L^2}^2 \leq \frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2 + \frac{\upsilon}{2}\left\|\nabla\vec{u}\right\|_{L^2}^2$$

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{u}\right\|_{L^2}^2 + \frac{\upsilon}{2}\left\|\nabla\vec{u}\right\|_{L^2}^2 \leq \frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2$$

By Poincaré inequality

$$\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 \le \frac{cL^2}{\upsilon}\|\vec{f}\|_{L^2}^2$$

By Gronwall's inequality

$$\|\vec{u}(t)\|_{L^2}^2 \le e^{-c\upsilon L^{-2}t}\ \|\vec{u}(0)\|_{L^2}^2 + \frac{cL^4}{\upsilon^2}\left(1-e^{-c\upsilon L^{-2}t}\right)\ \|\vec{f}\|_{L^2}^2 \quad \forall t\in[0,\mathrm{T}]$$

and

$$\upsilon\int_0^{\mathrm{T}}\|\nabla\vec{u}(\tau)\|_{L^2}^2\,d\tau \le K(L,\|\vec{u}_0\|_{L^2},\|\vec{f}\|_{L^2},\upsilon,\mathrm{T})$$

Theorem (Leray 1932-34)

For every $T > 0$ there exists a weak solution (in the sense of distribution) of the Navier-stokes equations, which also satisfies

$$\vec{u} \in C_w([0,T], L^2(\Omega)) \cap L^2([0,T], H^1(\Omega))$$

The question of uniqueness of weak solutions for the three-dimensional Navier-Stokes equations is still open.

Strong Solutions of Navier-Stokes

$$\vec{u} \in C([0,T], H^1(\Omega)) \cap L^2([0,T], H^2(\Omega))$$

Enstrophy

$$\|\nabla \times \vec{u}\|_{L^2}^2 = \|\vec{\omega}\|_{L^2}^2 = \|\nabla \vec{u}\|_{L^2}^2$$

Formal Enstrophy Estimates

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\vec{u}\right\|_{L^2}^2+\upsilon\left\|\Delta\vec{u}\right\|_{L^2}^2+\int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})+\int\nabla p(-\Delta\vec{u})=\int\vec{f}\cdot(-\Delta\vec{u})$$

Observe that $\int\nabla p\cdot(-\Delta\vec{u})dx=0$

By Cauchy-Schwarz $\left|\int\vec{f}\cdot(-\Delta\vec{u})\right|\le\frac{\left\|\vec{f}\right\|_{L^2}^2}{\upsilon}+\frac{\upsilon}{4}\left\|\Delta\vec{u}\right\|_{L^2}^2$

Nonlinear Estimates

One needs to find estimates for the nonlinearity:

$$\left|\int (\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})\right| \le ?$$

For example by Hőlder inequality

$$\left|\int (\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})\right| \le \|\vec{u}\|_{L^4}\ \|\nabla\vec{u}\|_{L^4}\ \|\Delta\vec{u}\|_{L^2}$$

Calculus/Interpolation (Ladyzhenskaya) Inequalities

$$\|\varphi\|_{L^4} \le \begin{cases} c\,\|\varphi\|_{L^2}^{1/2}\ \|\nabla\varphi\|_{L^2}^{1/2} & 2-D \\ c\,\|\varphi\|_{L^2}^{1/4}\ \|\nabla\varphi\|_{L^2}^{3/4} & 3-D \end{cases}$$

Denote by $y = e_0 + \|\nabla\vec{u}\|_{L^2}^2$

The Two-dimensional Case

$$\dot{y} \leq c\, y^2 \qquad \& \qquad \int_0^{\mathrm{T}} y(\tau)\, d\tau \leq K(\mathrm{T})$$

$$\Rightarrow y(t) \leq \tilde{K}(\mathrm{T})$$

Global regularity of strong solutions to the two-dimensional Navier-Stokes equations.

Navier-Stokes Equations

- Two-dimensional Case

  * Global Existence and Uniqueness
    of weak and strong solutions

  * Finite dimension global attractor

The Three-dimensional Case

Recall that $y = e_0 + \|\nabla \vec{u}\|_{L^2}^2$

One can show that
$$\dot{y} \le c(\|u\|_{L^6}^4 + e_0^2)\,y$$

Which implies that

$$y(t) \le y(0)\, e^{c\int_0^t (\|u(\tau)\|_{L^6}^4 + e_0^2)\,d\tau}$$

The Question Is Again Whether:

$$\int_0^T \|u(\tau)\|_{L^6}^4 \, d\tau \leq K ?$$

* Which is the $1 Million Question!!

* Or the -12 Bulls/Cows!!

Foias-Ladyzhenskaya-Prodi-Serrin Conditions

A strong solution of the three-dimensional Navier-Stokes equations exists on the interval [0,T] if and only if

$$u \in L^p((0,T), L^q(\quad)) \quad \text{for} \quad \frac{2}{p} + \frac{3}{q} = 1$$

For $1 \leq p \leq \infty.$

The case of $p=\infty$ and $q=3$ has been established by L. Escauriaza and G. Seregin and V. Sverak.

One can instead use the following Sobolev inequality

$$\|\vec{u}\|_{L^6} \le c\|\nabla\vec{u}\|_{L^2}$$

Which leads to $\dot{y} \le cy^3 \quad \& \quad \int_0^T y(\tau)d\tau \le K$

Theorem (Leray 1932-1934)

There exists $T_*(\|\vec{u}_0\|_{H^1}, \|\vec{f}\|_{L^2}, \upsilon, L)$ such that

$y(t) < \infty$ for every $t \in [0, T_*)$.

Navier-Stokes Equations

- The Three-dimensional Case

  * Global existence of the weak solutions

  * Short time existence of the strong solutions

  * Uniqueness of the strong solutions

- Open Problems:

  * Uniqueness of the weak solution.

  * Global existence of the strong solution.

Vorticity Formulation

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla)\vec{\omega} - \underline{\underline{(\vec{\omega} \cdot \nabla)\vec{u}}} = \nabla \times \vec{f}$$

Vorticity Stretching Term $(\vec{\omega} \cdot \nabla)\vec{u}$

Two dimensional case $(\vec{\omega} \cdot \nabla)\vec{u} \equiv \vec{0}$

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla)\vec{\omega} = \nabla \times \vec{f}$$

$\left|\vec{\omega}(x,t)\right|^2$ Satisfies a maximum principle.

The Three-dimensional Case

$$(\vec{\omega} \cdot \nabla)\vec{u} \not\equiv 0$$

$$\vec{\omega} \sim \mathrm{z}$$

$$(\vec{\omega} \cdot \nabla)\vec{u} \sim \mathrm{z}^2$$

For large initial data $\vec{\omega}_0$ the vorticity balance takes the form

$$\dot{\mathrm{z}} \sim \mathrm{z}^2 \Rightarrow$$ **Potential "Blow Up"!!**

Euler Equations ν=0

Theorem (Lechtenstein-1925): *Let* $u_0 \in C^{1,\alpha}$, *then there exists* $T_*(u_0) > 0$, *where the solution of the 3D Euler equations exists and unique, and* $u \in C([0, T_*], C^{1,\alpha})$.

Theorem (Ebin-Marsden,Kato,Temam,…): *Let* $u_0 \in H^s$, *for* $s > 5/2$, *then there exists* $T_*(u_0)$, *where the solution of the 3D Euler equations exists and unique, and* $u \in C([0, T_*], H^s)$.

Question: Does there exists a global weak solution for the 3D Euler equations?

Answer: YES (Wiedemann –February 2011)

- **Beale-Kato-Majda**

If $\int_0^T \|\vec{\omega}(t)\|_{L^\infty} \, dt < \infty$ then we have existence and uniqueness on the interval $[0, T]$

- That is, one has to “control” the $\|\vec{\omega}(t)\|_{L^\infty}$ in some way!!

- **Constantin, Fefferman and Majda**:

Provided sufficient condition involving the Lipschitz regularity of the direction of the vorticity:

$$\vec{\xi} = \frac{\vec{\omega}}{|\vec{\omega}|}$$

Weak Solutions for 3D Euler

- Existence of family (non-uniqueness) of weak solutions to the Cauchy problem of the 3D Euler has been recently proved by Wiedemann (2011).

- DeLellis – Szekelyhidi showed the existence of a non-trivial family of weak solutions to the 3D Euler equations with compact support in space and time.

The present proof is not a traditional PDE proof.

It shares many ideas of the proof of the [Nash (1974), Kuiper (1955)] theorem of invariant imbeddings of surfaces. It uses the formalism of differential inclusions, convex integration and accumulation of oscillations.

Earlier results were established by Shnirelman and Sheffer.

Ill-posedness of 3D Euler

Theorem (Bardos-Titi 2010):

(i) Let $u_1, u_3 \in L^2_{\text{loc}}(\mathbb{R})$ *then the shear flow*

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))$$

is a weak solution of the Euler equations, in the sense of distribution, in $= \mathbb{R}^3$.

(ii) Let $u_1, u_3 \in L^2_{\text{per}}(\mathbb{R})$ *then the shear flow above is a weak solution of the Euler equations, in the sense of distribtions. Furthermore, in this case the energy of this solution is constant.*

(iii) There exist shear flow solutions of the above form which, for $t = 0$, *belong to* $C^{0,\alpha}$, *for some* $\alpha \in (0,1)$, *and for* $t \neq 0$, *they do not belong to* $C^{0,\beta}$ *for any* $\beta > \alpha^2$.

This shear flow was used earlier by DiPerna and Majda to show that weak limit of oscillatory classical solutions of Euler equations does not converge to a solution of Euler equations.

Ruling Out Certain Weak Solutions of Euler

The work of DeLellis – Szekelyhidi implies that there are non-unique weak solutions of Euler even with very smooth initial data (for example $u_0 = 0$).

Question: Is there a natural criterion for ruling out certain weak solutions of Euler?

Ruling Out Criterion

In the absence of physical boundaries

A weak solution of Euler which cannot be achieved as a limit of Navier-Stokes, as the viscosity tends to zero, should be ruled out.

Vanishing Viscosity Solutions

[Bardos, Titi, Wiedemann, 2012]

For initial data of the form $(u_1(x_2), 0, u_3(x_1))$ there are infinitely many weak solutions of Euler equations.

The shear flow solution

$$(u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$$

is the vanishing viscosity limit solution.

Two-Dimensions Euler

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = 0$$

$$\vec{u} = \nabla \times (\psi \vec{k})$$

$$\Delta \psi \, \vec{k} = \vec{\omega}$$

- Yudovich proved a weak version of the maximum principle, that is $\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}$.

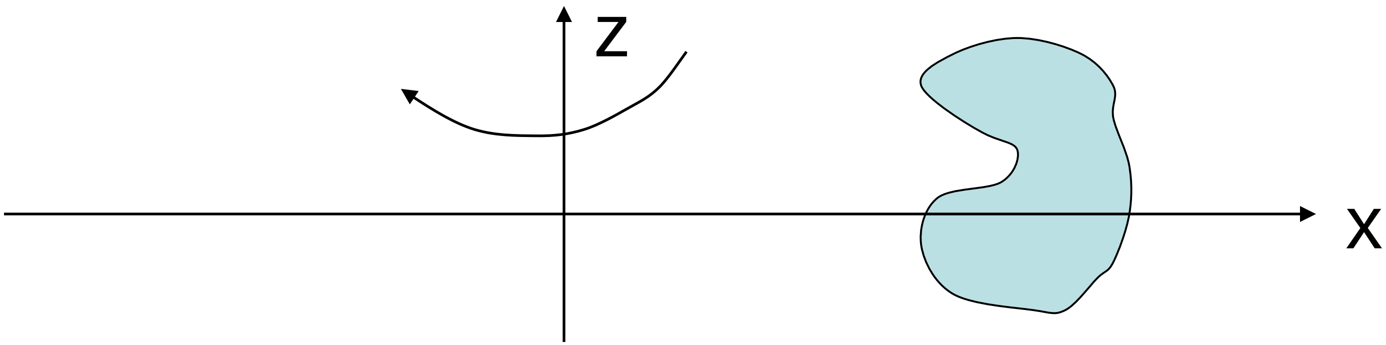
The idea of for the uniqueness

$$\|\psi\|_{W^{2,p}} = \sum_{|\alpha|\leq 2} \|D^\alpha \psi\|_{L^p} \leq c \cdot p \|\Delta \psi\|_{L^p}$$

Special Results of Global Existence for the three-dimensional Navier-Stokes

Theorem (Fujita and Kato)

Let $\|u_0\|_{H^{1/2}}$ be small enough. Then the 3D Navier-Stokes equations are globally well-posed for all time with such initial data. The same result holds if the initial data is small in $L^3(\Omega)$ (Kato, Giga & Miyakawa)



• Ω – Revolution Domain around the z-axis

[away from z-axis]

• Let us move to Cylindrical coordinates

Theorem (Ladyzhenskaya) Let

$$\vec{u}_0(x,y,z) = (\varphi_r^0(r,z), \varphi_\theta^0(r,z), \varphi_z^0(r,z))$$

be axi-symmetric initial data. Then the three-dimensional Navier-Stokes equations have globally (in time) strong solution corresponding to such initial data. Moreover, such strong solution remains axi-symmetric.

Theorem (Leiboviz, Mahalov and E.S.T.)

Consider the three-dimensional Navier-Stokes equations in an infinite Pipe. Let

$$\vec{u}_0 = (\varphi_r^0(r, n\theta + \alpha z), \varphi_\theta^0(r, n\theta + \alpha z), \varphi_z^0(r, n\theta + \alpha z))$$

(Helical symmetry). For such initial data we have global existence and uniqueness. Moreover, such a solution remains helically symmetric.

Remarks

For axi-symmetric and helical flows the vorticity stretching term is nontrivial, and the velocity field is three-dimensional.

In the inviscid case, i.e. $\upsilon = 0$, the question of global regularity of the three-dimensional helical or axi-symmetrical Euler equations is still open. Except the invariant sub-spaces where the vorticity stretching term is trivial.

More about vorticity stretching

In the two-and-half dimensional Navier-Stokes and Euler equations (u,v,w)(x,y).

(u,v) satisfy the 2D Navier-Stokes/Euler equations, and w is a passive scalar

The vorticity stretching term is non-trivial.

$$(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, 0) \cdot (\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, 0) = \frac{\partial w}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial y}$$

Does 2D Flow Remain 2D?

[Bardos, Lopes-Filho,Nussenzveig-Lopes, Niu, Titi (2012)]

- Let u_0 be a function of on (x,y), then the Leray-Hopf weak solution of the 3D Navier-Stokes remains a function of only (x,y).
- Similar result holds for the Navier-Stokes equations with axi-symmetric initial data, or helical initial data.
- Let u_0 be a function of (x,y), then the weak solution of the 3D Euler might become a function of (x,y,z).
- Also, if the initial data is axi-symmetric or helical symmetric, the weak solutions of Euler might break the symmetry.

- **Theorem [Cannone, Meyer & Planchon] [Bondarevsky] 1996**

Let M be given, as large as we want. Then there exists K(M) such that for every initial data of the form

$$\vec{u}_0 = \sum_{|\vec{k}| \geq K(M)} \hat{\vec{u}}^{\,0}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

[VERY OSCILLATORY]

satisfying $\|u_0\|_{H^1} \leq M$ *the three-dimensional Navier-Stokes equations have global existence of strong solutions.*

Remark Such initial data satisfies $\|u_0\|_{H^{1/2}} << 1.$

So, this is a particular case of Kato's Theorem.

Raugel and Sell (Thin Domains)

Global Existence (in time) of strong solutions for the 3D Navier-Stokes equations in thin domains, for large (depending on the thinness of the domain), initial data.

I. Kukavica and M. Ziane, made a significant improvement in the size of the data for which the above result holds, and investigated other physical configurations.

Stability of Strong Solutions

- [Ponce, Sideris, Racke & Titi, CMP 1994] The set of globally regular solutions of the 3D Navier-Stokes is stable under small perturbations in the initial data and forcing.

The Effect of Rotation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p + \vec{\Omega} \times \vec{u} = 0$$

$$\nabla \cdot \vec{u} = 0$$

- There is $\Omega_0(\mathrm{T}, \vec{u}_0)$ such that if $|\Omega| > \Omega_0$ the solution exists on $[0, \mathrm{T})$.
- That is there exists $\mathrm{T}_0\ (\vec{u}_{0,} |\vec{\Omega}|)$ such that the solution exists on $[0, \mathrm{T}_0)$.

Observe that $\mathrm{T}_0 \rightarrow \infty$ as $|\vec{\Omega}| \rightarrow \infty$

- Babin, Mahalov and Nicolaenko
- Embid and Majda
- Chemin, Desjardines, Gallagher, Grenier
- Masmoudi
- Ziane
- Liu and Tadmor

An Illustrative Example

Inviscid Burgers Equation

$$u_t + uu_x = 0 \quad \text{in } \mathbb{R}$$

$$u(x,0) = u_0(\mathrm{x})$$

- If $u_0(x)$ is decreasing function on some subinterval of $\mathcal{R}$ then the solution of the above equation develops a singularity (Shock) in finite time.

The solution is given implicitly by the relation:

$$u(x,t) = u_0(x - tu(x,t))$$

The Effect of the Rotation

$$u \in \mathrm{C} \qquad z \in \mathrm{C}$$

$$u_t + u u_z + i\Omega u = 0$$

$$u_0(z) = u(z,0)$$

$$v(z,t) = e^{i\Omega t} u(z,t)$$

$$v_t + e^{-i\Omega t} v v_z = 0$$

$$v(z,t) = v_0\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)$$

$$\frac{\partial}{\partial z} v = \frac{v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}{1 + \frac{e^{-i\Omega t} - 1}{-i\Omega} v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}$$

If $\Omega >> 1$, (i.e. $\Omega > \Omega_0(u_0)$)

$\frac{\partial}{\partial z} v$ remains finite and the solution remains regular for all $t \geq 0$.

Fast Rotation = Averaging

$$v(z,t) = v(z,0) - \int_0^t e^{-i\Omega t} v(z,t) v_z(z,t) dt$$

Riemann-Lebesgue Lemma

$$\lim_{k \to \infty} \int_0^{2\pi} \sin(kt)\varphi(t) dt = 0$$

The above complex system is equivalent to 2D Rotating Burgers:

$$u = u_1 + iu_2, \qquad z = x + iy$$

$$\vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \Omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{u} = 0$$

How can the computer help in solving the3D Navier-Stokes equations and turbulent flows?

Reynolds numbers = Ratio of the Intensity of the Nonlinear Effect to the Intensity of the Viscous Linear Effect.

Weather Prediction

In the atmosphere the Reynolds number is of the order of 7×10^8

In this case the number of equations needed to be solved at each time step is of the order of $Re^{9/4}$, i.e. 10^{18}.

Flow Around the Car

For the around a moving car the Reynolds number is of the order of 10^5

In this case the number of equations is of the order of 10^{10}

How long does it take to compute the flow around the car for a short time?

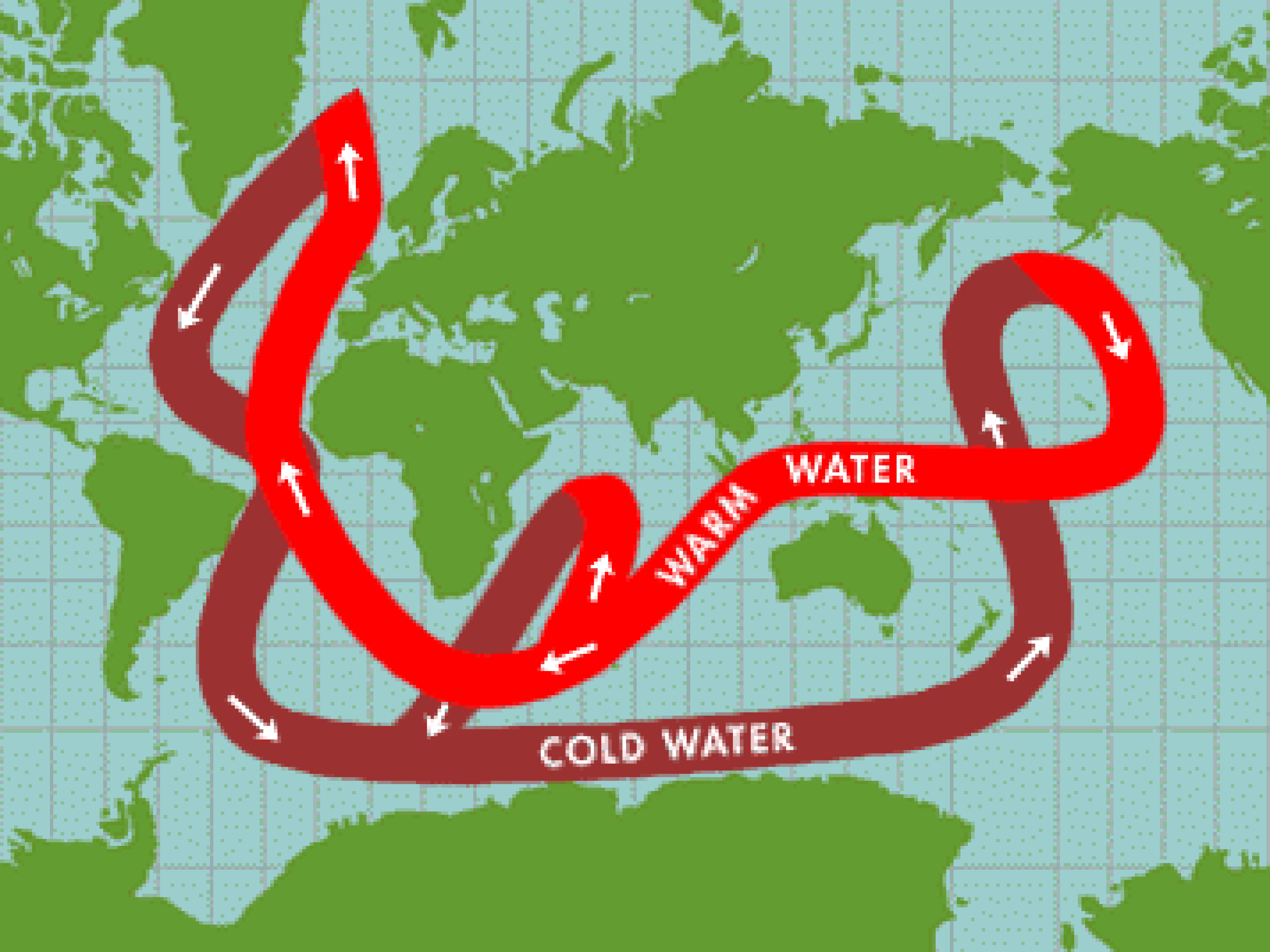
5 years?

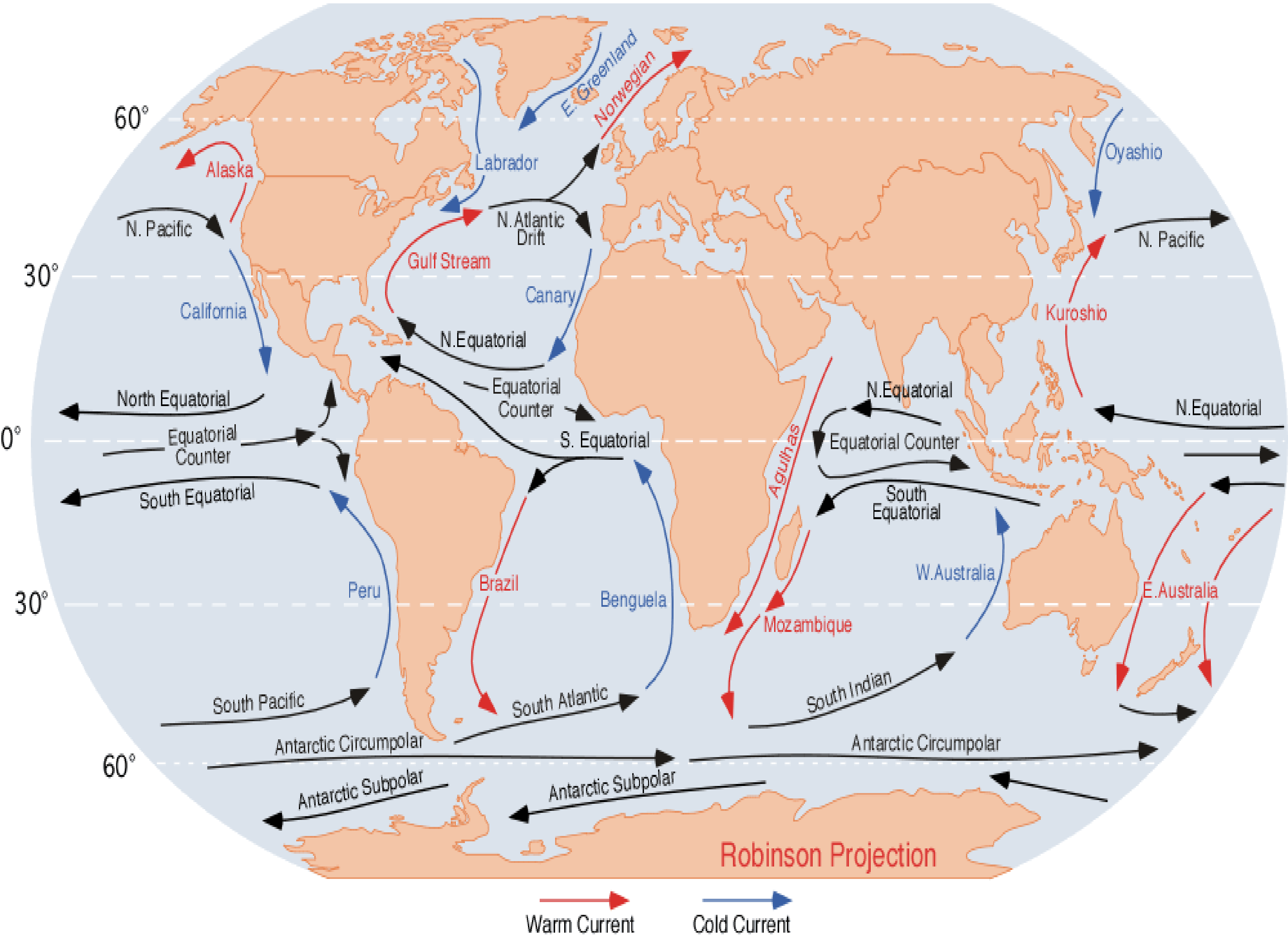
50 years?

500 years?

5000 years!!!

Large Scale Oceanic Circulations


WARM WATER
COLD WATER



60°
30°
0°
30°
60°
Alaska
N. Pacific
California
North Equatorial
Equatorial Counter
South Equatorial
Peru
South Pacific
Antarctic Circumpolar
Antarctic Subpolar
Labrador
E. Greenland
Norwegian
N. Atlantic Drift
Gulf Stream
Canary
N.Equatorial
Equatorial Counter
S. Equatorial
Brazil
Benguela
South Atlantic
Antarctic Subpolar
Agulhas
Mozambique
N.Equatorial
Equatorial Counter
South Equatorial
W.Australia
South Indian
Antarctic Circumpolar
Oyashio
N. Pacific
Kuroshio
N.Equatorial
E.Australia
Robinson Projection
Warm Current
Cold Current

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$$\frac{\partial}{\partial t}T - \kappa\Delta T + (\vec{u}\cdot\nabla)T = 0$$

Bénard Convection/Boussinesq Approximation

$$\frac{\partial}{\partial t}v_H - \upsilon\left(\Delta_H + \frac{\partial^2}{\partial z^2}\right)v_H + (v_H \cdot \nabla_H)v_H + w\frac{\partial}{\partial z}v_H + \frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = 0$$

$$\frac{\partial}{\partial t}w - \upsilon\left(\Delta_H + \frac{\partial^2}{\partial z^2}\right)w + (v_H \cdot \nabla_H)w + w\frac{\partial}{\partial z}w + \frac{1}{\rho_0}\frac{\partial}{\partial z}p + Tg = 0$$

$$\nabla_H \cdot v_H + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial t}T - \kappa\Delta T + (v_H \cdot \nabla_H)T + w\frac{\partial}{\partial z}T = \rho_0 Q$$

Here $(v_H, w) = \vec{u}.$

Typical Scales in the Ocean

- horizontal distance $L \sim 10^6$ m
- horizontal velocity $U \sim 10^{-1}$ m/s
- depth $H \sim 10^3$ m
- Coriolis parameter $f \sim 10^{-4}$ 1/s
- gravity $g \sim 10$ m/s^2
- density $\rho_0 \sim 10^3$ kg/m^3

Calculating the typical values

- Typical vertical velocity $$W = UH/L \sim 10^{-4}\ \mathrm{m/s}$$
- Typical pressure $$P = \rho_0\, g\, H \sim 10^{7}\ \mathrm{Pa}$$
- Typical time scale $$T = L/U \sim 10^{7}\ \mathrm{s}$$

Scale Analysis of Vertical Motion – The Ideal Case

$$\frac{\partial}{\partial t}w+(v_H\cdot\nabla_H)w+w\frac{\partial}{\partial z}w+\frac{1}{\rho_0}\frac{\partial}{\partial z}p+Tg=0$$

$$\frac{W}{\mathrm{T}}+\frac{UW}{L}+\frac{W^2}{H}+\frac{P}{H\rho_0}+Tg=0$$

$$10^{-11}+10^{-11}+10^{-11}+10+10=0$$

Hydrostatic Balance

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p + Tg = 0$$

The Primitive Equations of Large Scale Oceanic and Atmospheric Dynamics

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \nabla_H p + f \vec{k} \times v_H$$
$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p + gT = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = Q + K_h \Delta_H T + K_v T_{zz}$$

- Introduced by Richardson (1922)

  For Weather Prediction

- J.L. Lions, R. Temam, S. Wang (1992)

Gave Some Asymptotic Derivation of the Model.

Previous Results

- J.L. Lions, Temam, S. Wang (1992), and Temam, Ziane (2003)
The global existence of the weak solutions (No Uniqueness).

- Guillen-Gonzalez, Masmoudi, Rodriquez-Bellido (2001), and Temam, Ziane (2003)
The short time existence of the strong solution

- Temam, Ziane (2003)
Global Existence of Strong Solution for the 2-D case.

- C. Hu, Temam, Ziane (2003)
Global Regularity for Restricted (Large) Initial Data in Thin Domains.

Results

- C. Cao and E.S.T. [*Annals of Mathematics* (2007)]. [arXiv March 1, 2005].

  * the global existence of the weak solutions (Galerkin method/announced)

  * the global existence and uniqueness of the strong solutions.

  * existence of the global attractor (announced).

  * See also, [Ning Ju (2007)] for the existence of global attractor.

The Primitive Equations of Large Scale Oceanic and Atmospheric Dynamics

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \nabla_H p + f \vec{k} \times v_H$$
$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p + gT = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = Q + K_h \Delta_H T + K_v T_{zz}$$

A different formulation of the PE

$$w(x,y,z) = -\int_{-h}^{z} \nabla_H \cdot v_H(x,y,\xi)\, d\xi$$

$$p(x,y,z) = p_s(x,y) - g\int_{-h}^{z} T(x,y,\xi)\, d\xi$$

$$\bar{v}_H(x,y) = \tfrac{1}{h}\int_{-h}^{0} v_H(x,y,\xi)\, d\xi\,, \quad \nabla_H \cdot \bar{v}_H = 0$$

$$\tilde{v}_H(x,y,z) = v_H(x,y,z) - \bar{v}_H(x,y)$$

The Barotropic Mode – The Averaged Part of the Horizontal Velocity

$$\partial_t \bar{v}_H + \overline{(v_H \cdot \nabla_H) v_H + w \partial_z v_H} + f \vec{k} \times \bar{v}_H + \nabla_H p_s$$

$$= A_h \Delta_H \bar{v}_H + \nabla_H \overline{\int_{-h}^{z} gT \, dz}$$

The Baroclinic Mode –The Fluctuation Part of the Horizontal Velocity

$$\partial_t \tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\bar{v}_H + (\bar{v}_H \cdot \nabla_H)\tilde{v}_H +$$

$$\left(-\int_{-h}^{z} \nabla_H \cdot v_H \, dz\right)\partial_z \tilde{v}_H + f\,\vec{k} \times \tilde{v}_H -$$

$$\overline{(\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\nabla_H \cdot \tilde{v}_H)\tilde{v}_H} =$$

$$A_h \Delta_H \tilde{v}_H + A_v \partial_{zz} \tilde{v}_H + \nabla_H \int_{-h}^{z} gT \, d\xi - \nabla_H \overline{\int_{-h}^{z} gT \, d\xi}$$

The IDEA – Focus on Burgers Equation

$$u_t - \nu \Delta u + (u \cdot \nabla) u = 0$$

We have

$$\frac{1}{2}\partial_t |u(x,t)|^2 - \frac{1}{2}\Delta |u(x,t)|^2 + \sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{1}{2} u \cdot \nabla |u(x,t)|^2 = 0$$

A maximum principle for $|u(x,t)|^2$ and L^∞ bound.

Global Regularity for 1D, 2D and 3D Burgers Equation.

The Pressure Term!!

- Is the major difference between Burgers and the Navier-Stokes equations.
- What about in our system?

The Averaged Equation is "like" the 2D Navier-Stokes.

$$\partial_t \bar{v}_H + \overline{(v_H \cdot \nabla_H) v_H + w \partial_z v_H} + f \vec{k} \times \bar{v}_H + \nabla_H p_s$$

$$= A_h \Delta_H \bar{v}_H + \nabla_H \overline{\int_{-h}^{z} gT \, dz}$$

Where $p_s(x, y)!!$

The Fluctuation Equation is "like" 3D Burgers Equations – Has No Pressure Term!!

$$\partial_t \tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\bar{v}_H + (\bar{v}_H \cdot \nabla_H)\tilde{v}_H +$$

$$\left(-\int_{-h}^{z} \nabla_H \cdot v_H \, dz\right)\partial_z \tilde{v}_H + f\,\vec{k} \times \tilde{v}_H -$$

$$\overline{(\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\nabla_H \cdot \tilde{v}_H)\tilde{v}_H}$$

$$= A_h \Delta_H \tilde{v}_H + A_v \partial_{zz} \tilde{v}_H + \nabla_H \int_{-h}^{z} gT \, d\xi - \nabla_H \overline{\int_{-h}^{z} gT \, d\xi}$$

A-priori Estimates

- $\left\| \tilde{v}_H \right\|_{L^6} \le K$
- $\left\| \nabla_H \bar{v}_H \right\|_{L^2} \le K$

$\Rightarrow \left\| \bar{v}_H \right\|_{L^6} \le K$

$\Rightarrow \left\| v_H \right\|_{L^6} = \left\| \bar{v}_H + \tilde{v}_H \right\|_{L^6} \le K$

Q.E.D.

One of the Main Estimates Used

By Ladyzhenskaya inequality we usually have:

$$\left| \int_{\Omega} \left[h(x,y,z)\, f(x,y,z)\;\; g(x,y,z) \right] dxdy\, dz \right|$$

$$\leq C \| h \|_{L^2(\Omega)}^{1/4} \| h \|_{H^1(\Omega)}^{3/4} \| f \|_{L^2(\Omega)}^{1/4} \| f \|_{H^1(\Omega)}^{3/4} \| g \|_{L^2(\Omega)}$$

But here instead we have [Cao-Titi]:

$$\left| \int_{\Omega} \left[\left(\int_{-h}^{0} u(x,y,z)\, dz \right) f(x,y,z)\;\; g(x,y,z) \right] dxdy\, dz \right|$$

$$\leq C \| u \|_{L^2(\Omega)}^{1/2} \| u \|_{H^1(\Omega)}^{1/2} \| f \|_{L^2(\Omega)}^{1/2} \| f \|_{H^1(\Omega)}^{1/2} \| g \|_{L^2(\Omega)}$$

Using this observation about the pressure that it is effectively a function of two variables:

I. Kukavica and M. Ziane, (Nonlinearity 2007) obtained similar results for the case of Dirichlet boundary conditions, on top and bottom, and periodic boundary conditions in the horizontal direction.

Back to the 3D Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0} \nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

New Criterion for Global Regularity of the 3D Navier-Stokes Equations

Theorem (C. Cao and E.S.T. 2008):

The strong solution of the 3D Navier-Stokes equations exists on the interval [0, T] for as long as

$$\partial_z p \in L^r((0,T), L^s(\Omega)), \quad \frac{2}{r} + \frac{3}{s} \leq \frac{20}{7}, \text{ where } \; r \geq 1 \text{ and } s \geq \frac{21}{16}.$$

This is different than the result of L. Berselli and G.P. Galdi (2002) and of Y. Zhou (2005) where the assumption is on ∇p.

Recently this result was improved by Zhou and Pokorny

- There are other global regularity criteria involving the pressure due to:

Chae

Kukavica

and

Struwe.

- Sereigen and Sverak:

Global regularity if the pressure is bounded from below.

Most Recent Criterion for Global Regularity

Theorem (Cao and T.) [Arch. Rational Mech. Anal. 2011]

The three-dimensional Navier-Stokes equations has a unique storng solution on the interval $[0,T]$ *if and only if for some* $j,k \in \{1,2,3\}$

$$\frac{\partial u_j}{\partial x_k} \in L^\beta([0,T], L^\alpha(I\!R^3)); \quad \text{when } k \neq j, \text{and where}$$

$$\alpha > 3, 1 \cdot \quad \beta < \infty, \text{and } \frac{3}{\alpha} + \frac{2}{\beta} \cdot \quad \frac{\alpha+3}{2\alpha},$$

or

$$\frac{\partial u_j}{\partial x_j} \in L^\beta([0,T], L^\alpha(I\!R^3)); \quad \text{where } \alpha > 2, 1 \cdot \quad \beta < \infty,$$

$$\text{and } \frac{3}{\alpha} + \frac{2}{\beta} \cdot \quad \frac{3(\alpha+2)}{4\alpha}.$$

The Primitive Equations for Stratified Fluid Flows

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \frac{1}{\rho_0} \nabla_H p + f \vec{k} \times v_H$$

$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p = gT$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = 0$$

Global Regularity of the Primitive Equations with vertical Diffusion Turbulence Mixing

Cao-Titi. [Comm. Math. Physics 2011]

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \frac{1}{\rho_0} \nabla_H p + f \vec{k} \times v_H$$

$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p = gT$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = \kappa_T T_{zz}$$

Global Regularity of the Primitive Equations with Vertical Diffusion Turbulence Mixing

Cao-Li-Titi [ARMA 2014]

- Improvement of the results with Vertical Diffusion Turbulence Mixing

Global Regularity of the Primitive Equations with Horizontal Diffusion Mixing

Cao-Li-Titi [JDE 2014]

- Global Regularity of PE with Horizontal Diffusion Turbulence Mixing

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \frac{1}{\rho_0} \nabla_H p + f \vec{k} \times v_H$$

$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p = gT$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = \kappa_h \Delta_H T$$

Global Regularity of the Primitive Equations with Horizontal Viscosity & Diffusion Mixing

Cao-Li-Titi [2014]

- Global Regularity of PE with Horizontal Viscosity and Horizontal Diffusion

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \frac{1}{\rho_0} \nabla_H p + f \vec{k} \times v_H$$

$$= A_h \Delta_H v_H$$

$$\partial_z p = gT$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = \kappa_h \Delta_H T$$

Generalization of the Brezis-Gallouet inequality

[Cao-Li-Titi-2014]

Let $F \in W^{1,p}(\Omega)$, with $p > 3$, be a periodic function. Then

$$\|F\|_\infty \leq C_{p,\lambda} \max\left\{1, \sup_{r\geq 2} \frac{\|F\|_r}{r^\lambda}\right\} \log^\lambda(\|F\|_{W^{1,p}(\Omega)} + e).$$

Gronwall's Inequality for Systems

Let $m(t), K(t), A_i(t), B_i(t) \geq 0$ s.t. $A_i \geq e$, for $i = 1, \cdots, n$ with $K \in L^1_{\text{loc}}([0,\infty))$, and $m(t) \leq K(t) \log \sum_{i=1}^n A_i(t)$. Suppose that

$$\frac{d}{dt} A_1(t) + B_1(t) \leq m(t) A_1(t), \tag{1}$$

$$\frac{d}{dt} A_i(t) + B_i(t) \leq m(t) A_i(t) + \zeta A_{i-1}^{\alpha}(t) B_{i-1}(t), \quad i = 2, \cdots, n, \tag{2}$$

for any $t \in (0, \mathcal{T})$, where $\alpha \geq 1$ and $\zeta \geq 1$ are two constants. Then

$$\sum_{i=1}^n A_i(t) + \sum_{i=1}^n \int_0^t B_i(s)\,ds \leq Q(t), \quad \forall t \in [0, \mathcal{T}),$$

where Q is a continuous function on $[0,\infty)$, which is given explicitly in terms of $A_i(0), i = 1, \cdots, n$, and K.

Blowup of Inviscid Primitive Equations

Cao-Ibrahim-Nakanishi-Titi (2012) [Comm. Math Phys. 2013], see also Wong (2012)]

$$u_t + uu_x + vu_y + wu_z + p_x = 0,$$

$$v_t + uv_x + vv_y + wv_z + p_y = 0,$$

$$p_z + T = 0,$$

$$T_t + uT_x + vT_y + wT_z = \kappa_H \Delta_H T + \kappa_3 T_{zz},$$

$$u_x + v_y + w_z = 0.$$

In the horizontal channel

$$= \{(x, y, z) : 0 \cdot \quad z \cdot \quad H,\ (x, y) \in \mathbb{R}^2\}$$

Reduced Inviscid Primitive Equations

We take $v_0 = 0$ and $T_0 = 0$. This implies that $v \equiv 0$ and $T \equiv 0$. This yields the reduced 2D system

$$u_t + uu_x + wu_z + p_x = 0,$$

$$p_z = 0,$$

$$u_x + w_z = 0.$$

in the strip

$$M = \{(x, z) : 0 \cdot \quad z \cdot \quad H\,, x \in I\!R\}.$$

Boundary Conditions and the Pressure

$$w\big|_{z=H} = w\big|_{z=0} = 0$$

u, p and w are periodic in the x-direction, with period L.

u and p are odd functions of x, and w is an even function.

$$p_x(x,t) = \frac{-2}{H}\int_0^H u(x,z,t)u_x(x,z,t)\,dz$$

The Reduced System

$$u_{xt} + (uu_x)_x + w_x u_z + w u_{xz} - \frac{2}{H}\int_0^H (uu_x)_x \, dz = 0$$

$$u_x + w_z = 0$$

Since u is an odd function and and w is and even with respect to x we have:

$$u(0, z, t) = w_x(0, z, t) = 0$$

The Reduced Equation at x=0

Denote by $W(z,t) = w(0,z,t)$ then

$$W_{tz} - (W_z)^2 + WW_{zz} + \frac{2}{H}\int_0^H (W_z)^2 \, dz = 0$$

Which blows up in finite time. [Childress-Ierley-Spiegel-Young, 1989]

Self-similar Blowup Solution

$$W(z,t) = \frac{\varphi(z)}{1-t}, \text{ with } \varphi(0) = \varphi(H) = 0$$

Then

$$\varphi' - (\varphi')^2 + \varphi\varphi'' + \frac{2}{H}\int_0^H (\varphi'(z))^2 dz = 0$$

Which has a non-trivial solutions.

Thank You!