Hamiltonian and quasi-Hamiltonian reduction via derived symplectic geometry

Pavel Safronov

University of Oxford

BMD, November 2014

Introduction

Derived symplectic geometry studies symplectic and *shifted* symplectic structures on (derived) stacks.

- **Examples**: pt/G, $T^*[n]X$, character stacks of compact manifolds, ...
- **Goal**: explain how Hamiltonian reduction fits into the framework.
- **Also**: quasi-Hamiltonian reduction, fusion, symplectic implosion etc have natural interpretations.

Hamiltonian reduction

Definition

A *G-Hamiltonian space M* is a symplectic manifold (M, ω) with a compatible G-action and a G-equivariant map $\mu\colon M \to \mathfrak{g}^*$ satisfying

$$\mathrm{d}_{\mathrm{dR}}\mu(v) = \iota_{a(v)}\omega$$

for every $v \in \mathfrak{g}$.

Reduced space:

$$M//G = \mu^{-1}(0)/G = (M \times_{\mathfrak{g}^*} \mathrm{pt})/G \cong M/G \times_{\mathfrak{g}^*/G} \mathrm{pt}/G.$$

Hamiltonian reduction

Reduced space:

$$M//G = \mu^{-1}(0)/G = (M \times_{\mathfrak{g}^*} \mathrm{pt})/G \cong M/G \times_{\mathfrak{g}^*/G} \mathrm{pt}/G.$$

One has

$$T^*(\mathrm{pt}/G) = \mathfrak{g}^*[-1]/G,$$

so

$$T^*[1](\mathrm{pt}/G) = \mathfrak{g}^*/G.$$

It is a *1-shifted symplectic stack*. M/G and pt/G are two Lagrangians and Lagrangian intersection is again symplectic.

Hamiltonian reduction

Reduced space:

$$M//G = \mu^{-1}(0)/G = (M \times_{\mathfrak{g}^*} \mathrm{pt})/G \cong M/G \times_{\mathfrak{g}^*/G} \mathrm{pt}/G.$$

Examples:

1. Let X be a G-space. Then T^*X is a G-Hamiltonian space. The reduction is

$$T^*X//G \cong T^*(X/G).$$

2. A coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is a G-Hamiltonian space. The symplectic structure is given by the Kirillov–Kostant–Souriau form.

Derived symplectic structures

Space = derived stack. $\mathbb{T}_X$ is a complex of bundles.

Definition

An *n-shifted symplectic structure* ω_X on a space X is an isomorphism $\omega_0\colon \mathbb{T}_X \xrightarrow{\sim} \mathbb{T}^*_X[n]$ together with some closedness data.

Really: a collection of differential forms $\omega_0, \omega_1, \ldots$ satisfying

$$d\omega_0 = 0, \mathrm{d}_{\mathrm{dR}}\omega_0 + d\omega_1 = 0, \ldots$$

Here ω_0 is a degree n two-form, ω_1 is a degree $n-1$ three-form and so on.
Remark: p-forms of degree q are similar to (p, q)-forms in the Dolbeault complex. d_{dR} is similar to ∂ and d is similar to $\overline{\partial}$.

Derived symplectic structures

Space = derived stack. $\mathbb{T}_X$ is a complex of bundles.

Definition

An *n-shifted symplectic structure* ω_X on a space X is an isomorphism $\omega_0\colon \mathbb{T}_X \xrightarrow{\sim} \mathbb{T}^*_X[n]$ together with some closedness data.

Examples:

- $T^*[n]X$ has a symplectic structure of degree n.
- pt/G is 2-shifted symplectic. $\mathbb{T}_{\mathrm{pt}/G} \cong \mathfrak{g}[1]$, $\mathbb{T}^*_{\mathrm{pt}/G} \cong \mathfrak{g}^*[-1]$.

Derived symplectic structures

Space = derived stack. $\mathbb{T}_X$ is a complex of bundles.

Definition

An *n-shifted symplectic structure* ω_X on a space X is an isomorphism $\omega_0 \colon \mathbb{T}_X \xrightarrow{\sim} \mathbb{T}^*_X[n]$ together with some closedness data.

- One can make sense of *isotropic* and *Lagrangian* morphisms $f \colon L \to X$ into an n-shifted symplectic space.
- An *isotropic structure* is a homotopy $f^*\omega_X \sim 0$. A morphism $L \to \mathrm{pt}$ is Lagrangian iff L is n-shifted symplectic.

Theorem (PTVV)

An intersection of two Lagrangians $L_1 \times_X L_2$ in an n-shifted symplectic space is $(n-1)$-shifted symplectic.

Hamiltonian reduction revisited

Recall that $\mathfrak{g}^*/G \cong T^*[1](\mathrm{pt}/G)$ is 1-shifted symplectic.
Given a G-space M and a G-equivariant map $\mu\colon M \to \mathfrak{g}^*$, when is $\mu\colon M/G \to \mathfrak{g}^*/G$ Lagrangian?
Need a degree 0 two-form h on M/G, such that

$$\mu^*\omega_0 = dh$$
$$0 = \mathrm{d}_{\mathrm{dR}} h.$$

That is, h is a G-invariant symplectic form on M satisfying the moment map equation.

Theorem (Calaque, S)

Lagrangians in $\mathfrak{g}^*/G$ are the same as G-Hamiltonian spaces.

Hamiltonian reduction revisited

Theorem (Calaque, S)

Lagrangians in $\mathfrak{g}^*/G$ are the same as G-Hamiltonian spaces.

$\mathrm{pt}/G \to \mathfrak{g}^*/G$ is also Lagrangian. Thus,

$$M_{red} \cong M/G \times_{\mathfrak{g}^*/G} \mathrm{pt}/G$$

is a Lagrangian intersection, so it carries an ordinary symplectic structure.

Symplectic implosion

Theorem (S)

Given a Lagrangian $L \to X$ and a Lagrangian correspondence $X \leftarrow N \to Y$, the morphism $L \times_X N \to Y$ is Lagrangian.

Let $B \subset G$ be a Borel subgroup and H the maximal torus.

Theorem (S)

The correspondence

$$\begin{array}{ccccc} & & \mathfrak{b}/B & & \\ & \swarrow & & \searrow & \\ \mathfrak{g}/G & & & & \mathfrak{h}/H \end{array}$$

is Lagrangian.

This allows one to turn G-Hamiltonian spaces into H-Hamiltonian spaces (a sort of abelianization). This procedure coincides with symplectic implosion of Guillemin, Jeffrey, Sjamaar.

AKSZ formalism

Let X be an n-shifted symplectic stack.

Theorem (PTVV)

Let M be a closed d-dimensional manifold. The stack of locally-constant maps $\mathrm{Map}(M_B, X)$ is $(n-d)$-shifted symplectic.

Theorem (Calaque)

Let M be a compact d-dimensional manifold. The restriction morphism

$$\mathrm{Map}(M_B, X) \to \mathrm{Map}((\partial M)_B, X)$$

is Lagrangian.

Example: since pt/G is 2-shifted symplectic,

$$\frac{G}{G} \cong \mathrm{Map}((S^1)_B, \mathrm{pt}/G)$$

is 1-shifted symplectic.

Quasi-Hamiltonian reduction

Let M be a G-space and $\mu\colon M \to G$ a G-equivariant map. When is $\mu\colon M/G \to \frac{G}{G}$ Lagrangian?
Need a degree 0 two-form on M/G, such that

$$\mu^*\omega_0 = dh$$
$$\mu^*\omega_1 = \mathrm{d}_{\mathrm{dR}} h.$$

Theorem (S)

Lagrangians in $\frac{G}{G}$ are the same as G-quasi-Hamiltonian spaces.

Quasi-Hamiltonian reduction is

$$M_{red} = \mu^{-1}(e)/G \cong M/G \times_{\frac{G}{G}} \mathrm{pt}/G,$$

which is again a Lagrangian intersection, hence symplectic.

Classical Chern-Simons theory

Given a manifold M, the phase space of classical Chern-Simons theory on M is $\mathrm{Map}(M_B, \mathrm{pt}/G)$, the space of G-local systems on M. The AKSZ theorems imply that $\mathrm{Map}(-, \mathrm{pt}/G)$ is a *classical topological field theory*: it sends closed manifolds to symplectic stacks and cobordisms to Lagrangian correspondences.

Example: a pair of pants gives a correspondence

$$\frac{G}{G} \times \frac{G}{G} \longleftarrow \frac{G \times G}{G} \longrightarrow \frac{G}{G}$$

Theorem (S)

Given two Lagrangians in $\frac{G}{G}$, composition with this correspondence produces another Lagrangian in $\frac{G}{G}$. This coincides with fusion of quasi-Hamiltonian spaces.