Commutators of singular integrals with BMO functions

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Ikerbasque

and

University of the Basque Country

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Societat Catalana de Matemàtiques

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This lecture is dedicated to the memory of my PhD advisor

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Björn Jawerth

November 25,1952 - September 2, 2013

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• **Rough homogeneous** singular integrals can be considered as well.

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• Operator theory: Hankel operator, Bergman spaces.

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Let $1 < p < \infty$. Then

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Theorem If T is any Calderón-Zygmund operator and if $b \in BMO(\mathbb{R}^n)$, then for any $1 < p < \infty$

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C. Fefferman-Stein($\approx$ 1973)

Let $0 < p < \infty$ and let $w \in A_\infty$. Then

$$\|M(f)\|_{L^p(w)} \leq c \, \|M^{\#}(f)\|_{L^p(w)}$$

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with sharp exponent $\frac{1}{p-1}$ which cannot be replaced by $\frac{1}{p-1} - \epsilon$

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$$\|M\|_{L^p(\mathbb{R}^n)} \approx \frac{1}{p-1} \qquad p \to 1$$

Muckenhoupt and Buckley's theorem.

• Recall the **Hardy–Littlewood maximal function**

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy$$

Theorem Let $1 < p < \infty$, then

$$M : L^p(w) \longrightarrow L^p(w) \quad \Longleftrightarrow \quad w \in A_p$$

Furthermore,

$$\|M\|_{L^p(w)} \leq c_{p,n}\,[w]_{A_p}^{\frac{1}{p-1}}$$

with sharp exponent $\frac{1}{p-1}$ which cannot be replaced by $\frac{1}{p-1} - \epsilon$

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Joint work with T. Luque and E. Rela.

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together with Muckenhoupt's theorem and the R.H.I.'s property of A_p weights.

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is **false** when b is a BMO function.

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where as above $\Phi(t) = t(1 + \log^+ t)$. In fact is **false** for $\Phi(t) = t$.

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Is sharp, M^2 cannot be replaced by M.

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The method of proof, by sharpening the **conjugation** method T_z

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If $w \in A_\infty$ satisfies a **Reverse Hölder Inequality**: for some $r, c > 1$

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Joint work with T. Hytönen.

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- The proof by Karagulyan is not so clear.

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Theorem (higher order case) Idem as above

$$\frac{1}{|Q|}\left|\left\{x \in Q : \frac{|T^k_b f(x)|}{M^{k+1} f(x)} > t\right\}\right| \le c\, e^{-(ct)^{1/(k+1)}} \qquad t > 0$$

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thank you