A geometric characterization of planar Sobolev extension domains

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We study those planar domains $\Omega \subset \mathbb{R}^2$ for which there exists an extension operator $T\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. Here the Sobolev space $W^{1,p}$, $1 \leq p \leq \infty$, is

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \ : \ \nabla u \in L^p(\Omega, \mathbb{R}^2) \right\},$$

where ∇u denotes the distributional gradient of u. The usual norm in $W^{1,p}(\Omega)$ is $||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}$. More precisely, $T\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ is an extension operator if there exists a constant $C \geq 1$ so that for every $u \in W^{1,p}(\Omega)$ we have

$$||Tu||_{W^{1,p}(\mathbb{R}^2)} \leq C||u||_{W^{1,p}(\Omega)}$$

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and $Tu|_\Omega = u$.
Notice that we are not assuming the operator T to be linear. However, for $p > 1$ there always exists also a linear extension operator provided that there exists an extension operator, see [9] and also [19]. Finally, a domain $\Omega \subset \mathbb{R}^2$ is called a $W^{1,p}$-extension domain if there exists an extension operator $T\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$.

We prefer to use the homogeneous norm $||u||_{L^{1,p}(\Omega)} = ||\nabla u||_{L^p(\Omega)}$. This makes no difference for us because we only consider domains Ω with bounded (and hence compact) boundary; for such domains one has a bounded (linear) extension operator for the homogeneous norms if and only for the non-homogeneous ones; see [11]. In what follows, the norm in question is always the homogeneous one, even if we happen to refer to it by $||u||_{W^{1,p}(\Omega)}$.

Jointly with Tapio Rajala and Yi Zhang we have very recently obtained the following geometric characterization of simply-connected bounded planar $W^{1,p}$-extension domains for $1 < p < 2$.

Theorem 1

Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. Then Ω is a $W^{1,p}$-extension domain if and only if for all $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining z_1 and z_2 such that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \leq C(\Omega, p)|z_1 - z_2|^{2-p}. \tag{1}$$

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$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) \leq C(\Omega, p)|z_1 - z_2|^{2-p}. \tag{1}$$

Both the necessity and sufficiency in Theorem 1 are new. Notice that the curve γ above is allowed to touch the boundary of Ω even if the points in question lie outside the closure of Ω. This is crucial: there exist bounded simply-connected $W^{1,p}$-extension domains for which $\mathbb{R}^2 \setminus \overline{\Omega}$ has multiple components, see e.g. [4].

When combined with earlier results, Theorem 1 essentially completes the search for a geometric characterization for bounded simply-connected planar $W^{1,p}$-extension domains. The unbounded case requires extra technical work and it will be discussed elsewhere. Theorem 1 leaves out the case $p = 1$ that requires additional arguments; we will deal with it in a subsequent paper.

The condition (1) on the complement in Theorem 1 appears also in the characterization of $W^{1,q}$-extension domains when $2 < q < \infty$. For such domains a characterization using the condition (1) in the domain itself with the Hölder dual exponent p of q was proved in [20, Theorem 1.2], see also earlier results [3, 14].

Theorem 2 (Shvartsman)

Let $2 < q < \infty$ and let Ω be a bounded simply-connected planar domain. Then Ω is a $W^{1,q}$ -extension domain if and only if for all $z_1, z_2 \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining z_1 to z_2 such that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-q}}\, ds(z) \leq C(\Omega, q)|z_1 - z_2|^{\frac{q-2}{q-1}}. \tag{2}$$

The above two theorems leave out the case $p = 2$. This is settled by earlier results [6, 7, 8, 12], according to which a bounded simply-connected domain is a $W^{1,2}$-extension domain if and only it is a quasidisk (equivalenty, a uniform domain). Since the complementary domain of a Jordan uniform domain is also uniform, one rather easily concludes that a Jordan domain is a $W^{1,2}$-extension domain if and only if the complementary domain is.

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Combining our characterization in Theorem 1 with Shvartsman's characterization stated in Theorem 2 one easily obtains the following duality result between the extendability of Sobolev functions from a Jordan domain and from its complementary domain.

Corollary 3

Let $1 < p, q < \infty$ be Hölder dual exponents and let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then Ω is a $W^{1,p}$-extension domain if and only if $\mathbb{R}^2 \setminus \bar{\Omega}$ is a $W^{1,q}$-extension domain.

Corollary 4

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected $W^{1,p}$-extension domain, where $1 < p \leq 2$. Then there is $q > p$ so that Ω is a $W^{1,s}$-extension domain for all $1 < s < q$.

Corollary 4

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected $W^{1,p}$-extension domain, where $1 < p \leq 2$. Then there is $q > p$ so that Ω is a $W^{1,s}$-extension domain for all $1 < s < q$.

This follows from the fact that (1) for $1 < p < 2$ implies the similar inequality for all $1 < s < p + \epsilon$. The case of smaller s is essentially just Hölder's inequality, see [17], while the improvement to larger exponents follows from the proof of Proposition 2.6 in [20]; consider a minimizer for (1) in $\mathbb{R}^2 \setminus \Omega$. Again, in the case $p = 2$, Corollary 4 was already known to hold: one then has extendability for all $1 < s < \infty$.

Combining Corollary 4 with results from [14] and [20] we obtain an open-ended property.

Corollary 5

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected $W^{1,p}$-extension domain, where $1 < p < \infty$. Then the set of all $1 < s < \infty$ for which Ω is a $W^{1,s}$-extension domain is an open interval.

Actually, the open interval above can only be one of $1 < s < \infty$, $1 < s < q$ with $q \leq 2$, or $q < s < \infty$ with $q \geq 2$.

Let us finally comment on some earlier partial results related to Theorem 1. First of all, it is well known that bounded simply-connected $W^{1,p}$-extension domains are John domains when $1 \le p < 2$, see [7, 18] and references therein. However, there exist John domains that fail to be extension domains and, even after Theorem 1 there is no interior geometric characterization available for this range of exponents.

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Secondly, in [15] it was shown that the complement of a bounded simply-connected $W^{1,1}$-extension domain is quasiconvex. This was obtained as a corollary to a characterization of bounded simply-connected BV-extension domains. Recall that a set $E \subset \mathbb{R}^2$ is called *quasiconvex* if there exists a constant $C \geq 1$ such that any pair of points $z_1, z_2 \in E$ can be connected to each other with a rectifiable curve $\gamma \subset E$ whose length satisfies $\ell(\gamma) \leq C|z_1 - z_2|$. In [15] it was conjectured that quasiconvexity of the complement holds for every $W^{1,p}$-extension domain when $1 \leq p \leq 2$. This conjecture follows from our Theorem 1, but again, quasiconvexity is a weaker condition than our geometric characterization.

Both the necessity and sufficiency are first proved for the approximating Jordan domains Ω_n obtained via $\varphi(B(0, 1 - 1/n))$, where $\varphi : \mathbb{D} \to \Omega$ is the conformal map (normalized so that $\varphi(0)$ is the John center of Ω).

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For necessity, one needs to know that the domains Ω_n are Sobolev-extension domains with a uniform bound on the norms of the extension operators. For this, one uses the fact that Ω is John and φ is quasisymmetric with respect to the internal metrics. Then Ω, Ω_n are uniform with respect to the internal metrics, and a variant of the extension method due to Jones allowss one to extend from Ω_n to Ω. For Ω_n, one then constructs suitable test-functions.

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Given a Whitney cube of the complementary domain of size no more than the diameter of our domain, the fact that Ω_n is John would give us a cube of comparable size at distance comparable to the size of our given Whitney cube.
This is not the way to construct the extension.

Fix a conformal map $\tilde{\varphi}$ from the complementary domain to the complementary domain of the unit disk. Given a Whitney cube $\tilde{Q}$ of the complementary domain, map it via $\tilde{\varphi}$. This gives us a shadow of on the circle, obtained via hyperbolic rays from infinity. Map this shadow back to the boundary of Ω_n via our complementary conformal map.

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Assign a Whitney cube of Ω_n to this shadow (via the interior conformal map) so that the internal shadow via hyperbolic rays is comparable to the given shadow.

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Pick a partition of unity and use averages from the "reflected cubes".
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Use the construction to "locate" the problematic cubes and (1) to control them.

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