Nonlinear Beltrami equations and quasiconformal flows

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Based in joint works with

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We say that $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is K-quasiregular if

* $f \in W^{1,2}_{loc}(\Omega; \mathbb{C})$, and
* f satisfies the **distortion inequality** with constant K,

$$|Df(z)|^2 \leq K\, J(z, f) \qquad a.e.$$

Equivalently,

$$|\partial_{\bar{z}} f(z)| \leq k\, |\partial_z f(z)| \qquad \text{a.e., with } k = \frac{K-1}{K+1},$$

or even

$$\partial_{\bar{z}} f(z) = \mu(z)\, \partial_z f(z) \qquad \text{a.e., for some } \mu : \ |\mu(z)| \leq k.$$

We say that f is K-quasiconformal if it is a KQR homeo.

$$K = 1 \Leftrightarrow k = 0 \Leftrightarrow \mu \equiv 0 \qquad \Rightarrow \begin{cases} KQR = \{\text{holomorphic}\} \\ KQC = \{\text{conformal}\} \end{cases}$$

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Measurable Riemann Mapping Theorem.
Given $a \neq 0$, $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \leq k < 1$,

$$\exists\,!\,\phi_a : \mathbb{C} \to \mathbb{C} \text{ such that } \begin{cases} \phi_a \text{ is } KQC \\ \partial_{\bar z}\phi_a = \mu\,\partial_z\phi_a & \leftarrow \textbf{Beltrami equation} \\ \phi_a(0) = 0, \phi_a(1) = a \end{cases}$$

Indeed,

$$\phi_a = a\,\phi_1$$

Therefore μ generates a set $\mathcal{F} = \{\phi_a\}_{a\in\mathbb{C}}$ such that

(1) $\phi_0 \equiv 0$

(2) if $a \neq 0$ then $\phi_a \in KQC$, $0 \mapsto 0$, $1 \mapsto a$

(3) $\mathcal{F}$ is stable under $\mathbb{C}$-linear combinations

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Fact: This also works conversely:

$$\mathcal{F} = \{\phi_a\}_{a \in \mathbb{C}} \text{ as before} \quad \rightsquigarrow \quad \phi_1$$

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Main reason: ϕ *KQC* $\Rightarrow$ $\partial_z \phi \neq 0$ almost everywhere

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Is $\mathbb{C}$-linearity important? NO!

Given $\mu, \nu \in L^\infty$ with $|\mu| + |\nu| \le k$,

$$\exists!\, \phi_a : \mathbb{C} \to \mathbb{C} \text{ such that } \begin{cases} \phi_a \text{ is } KQC \\ \partial_{\bar z}\phi_a = \mu\, \partial_z \phi_a + \nu\, \overline{\partial_z \phi_a} \\ \phi_a(0) = 0, \phi_a(1) = a \end{cases}$$

So now

$$\mathcal{F} = \{\phi_a\}_{a\in\mathbb{C}} \text{ is only } \mathbb{R}\text{-linear}$$

whence

$$\mathcal{F} = \{\alpha\, \phi_1 + \beta\, \phi_i\}_{\alpha,\beta\in\mathbb{R}}$$

Remark: In particular,

$$\phi_{\alpha+i\beta} = \alpha\phi_1 + \beta\phi_i$$

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$$\mathcal{F} = \{\phi_a\}_{a \in \mathbb{C}} \text{ as before} \quad \rightsquigarrow \quad \phi_1, \phi_i$$

$$\rightsquigarrow \quad \mu, \nu$$

Fundamental step: (Alessandrini, Nesi; Astala, Jääskeläinen)
If ϕ_1, ϕ_i are *KQC* and $\mathbb{R}$-linearly independent, then

$$\mathrm{Im}(\partial_z \phi_1 \, \overline{\partial_z \phi_i}) \neq 0 \qquad \text{almost everywhere.}$$

Indeed, under $\mathbb{C}$-linearity one has $\phi_i = i\,\phi_1$ and therefore

$$\mathrm{Im}(\partial_z \phi_1 \, \overline{\partial_z \phi_i}) = -|\partial_z \phi_1|^2$$

Conclusion: the pair μ, ν and $\mathcal{F}$ uniquely determine each other

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OUR GOAL: nonlinear counterpart

$\{$Nonlinear Beltrami equations$\} \rightleftarrows \{$nonlinear families$\}$

Nonlinear Beltrami equation (Bojarski, Iwaniec)

$$\partial_{\overline{z}} f = \mathcal{H}(z, \partial_z f)$$

where $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is such that:

(1) $\mathcal{H}(z, 0) = 0$

(2) $z \mapsto \mathcal{H}(z, w)$ is measurable

(3) $w \mapsto \mathcal{H}(z, w)$ is $k(z)$-Lipschitz, $\|k\|_\infty = \frac{K-1}{K+1} < 1$

Given one such $\mathcal{H}$, look for a family $\mathcal{F}_{\mathcal{H}} = \{\phi_a\}_{a \in \mathbb{C}}$

$$P_a : \begin{cases} \phi_a : \mathbb{C} \to \mathbb{C} \text{ is } KQC \\ \partial_{\overline{z}} \phi_a = \mathcal{H}(z, \partial_z \phi_a) \\ \phi_a(0) = 0, \phi_a(1) = a \end{cases}$$

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General Existence Theorem (Astala, Iwaniec, Martin)
If $\mathcal{H}$ is as before and $a \neq 0$, then P_a has always a solution ϕ_a.

Uniqueness theorem (ACFJS)
If $\mathcal{H}$ is as before and

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2}$$

then the solution to P_a , $a \neq 0$, is unique.
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Fact: TFAE:

- $\mathcal{H}$ has the uniqueness property
- f, g are KQC solutions of $\mathcal{H} \Rightarrow \; f - g$ bijective or constant

Corollary

If $\mathcal{H}$ has uniqueness then $\mathcal{F}_{\mathcal{H}}$ has KQC increments.

Proof:

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\begin{aligned}
|\partial_{\bar z}(\phi_a - \phi_b)| = |\mathcal{H}(z, \partial_z \phi_a) - \mathcal{H}(z, \partial_z \phi_b)| &\le k(z)\, |\partial_z(\phi_a - \phi_b)| \\
&\Rightarrow \phi_a - \phi_b \text{ is } KQR \\
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Conclusion : To every $\mathcal{H}$ with uniqueness we can associate a KQC flow $\mathcal{F}_{\mathcal{H}}$. If $w \mapsto \mathcal{H}(z, w)$ is linear then the flow $\mathcal{F}$ is linear too. Conversely?

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Given $\mathcal{F} = \{\phi_a\}_{a\in\mathbb{C}}$, find a unique $\mathcal{H} = \mathcal{H}_\mathcal{F}$ such that

$$\partial_{\bar z}\phi_a = \mathcal{H}(z, \partial_z \phi_a) \qquad \text{for a.e. } z \text{ and every } a$$

Roughly:
Given $(z, w) \in \mathbb{C} \times \mathbb{C}$,

1. Find $a = a(z, w)$ such that

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Automatically $\partial_{\bar z}\phi_a(z) = \mathcal{H}(z, \partial_z\phi_a(z))$ always. However,

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Measurability of $\mathcal{H}$? Lipschitz continuity of $\mathcal{H}$? Uniqueness of $\mathcal{H}$?

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Given $\mathcal{F} = \{\phi_a\}_{a \in \mathbb{C}}$, find a unique $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$ such that

$$\partial_{\bar{z}} \phi_a = \mathcal{H}(z, \partial_z \phi_a) \qquad \text{for a.e. } z \text{ and every } a$$

Roughly:
Given $(z, w) \in \mathbb{C} \times \mathbb{C}$,

1. Find $a = a(z, w)$ such that

$$\partial_z \phi_a(z) = w$$

2. Set

$$\mathcal{H}(z, w) = \partial_{\bar{z}} \phi_a(z).$$

Automatically $\partial_{\bar{z}} \phi_a(z) = \mathcal{H}(z, \partial_z \phi_a(z))$ always. However,

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Uniqueness of a: Assume that for (z, w) we have two solutions a_1, a_2 of the equation

$$\partial_z \phi_a(z) = w$$

Then

$$\partial_z(\phi_{a_1} - \phi_{a_2})(z) = \partial_z \phi_{a_1}(z) - \partial_z \phi_{a_2}(z) = 0$$

By quasiconformality of the increments,

$$\Rightarrow \partial_{\overline{z}}(\phi_{a_1} - \phi_{a_2})(z) = 0$$

$$\Rightarrow \partial_{\overline{z}}\phi_{a_1}(z) = \partial_{\overline{z}}\phi_{a_2}(z)$$

$$\Rightarrow \mathcal{H}(z, w) \text{ well defined}$$

Remark: exceptional sets

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Remark: exceptional sets

Existence of solutions: Given (z, w), can we expect the equation

$$\partial_z \phi_a(z) = w$$

to have at least one solution $a = a(z, w)$?

- $\mathbb{C}$-linear flows: $\phi_a = a\,\phi_1$ Thus

$$\Rightarrow \text{need to solve } a\,\partial_z\phi_1(z) = w$$
$$\Rightarrow \text{ it suffices } \partial_z\phi_1(z) \neq 0.$$

- $\mathbb{R}$-linear flows: $\phi_{\alpha+i\beta} = \alpha\,\phi_1 + \beta\,\phi_i$ and therefore

$$\Rightarrow \text{need to solve } \alpha\,\partial_z\phi_1(z) + \beta\,\partial_z\phi_i(z) = w$$
$$\Rightarrow \text{ it suffices } \operatorname{Im}(\partial_z\phi_1(z)\,\overline{\partial_z\phi_i(z)}) \neq 0.$$

- General $\mathcal{F} = \{\phi_a\}_{a\in\mathbb{C}}$: Full range condition

$$\{\partial_z\phi_a(z)\}_{a\in\mathbb{C}} = \mathbb{C} \qquad \text{a.e. } z \in \mathbb{C}.$$

Otherwise $\mathcal{H}_{\mathcal{F}}$ may be non unique for our given $\mathcal{F}$

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Key Lemma. If $\mathcal{F} = \{\phi_a\}_{a\in\mathbb{C}}$ smooth and non-degenerate then

$$a \mapsto \partial_z \phi_a(z)$$

is, for each fixed $z \in \mathbb{C}$, a global homeomorphism on $\mathbb{C}_\infty$.

Sketch of proof:

- *Smoothness*: $\begin{cases} z \mapsto \partial_z\phi_a(z) \text{cont, no exceptional set} \\ a \mapsto \partial_z\phi_a(z) \text{ continuous on } \mathbb{C} \end{cases}$
- *Non degeneracy* (I): for every fixed z,

$$\lim_{|a|\to\infty} |\partial_z\phi_a(z)| = \infty$$

- *Non degeneracy* (II): for every fixed z,

$$a \mapsto \partial_z\phi_a(z) \text{ locally injective on } \mathbb{C}$$

Topology: Given $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, continuous on $\mathbb{C}_\infty$ and locally injective on $\mathbb{C}_\infty \setminus \{p\}$, then f is a global homeomorphism.

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Is the key lemma realistic?

Second key lemma. If $\mathcal{H}$ is smooth and has the uniqueness property then $\mathcal{F}_{\mathcal{H}}$ is smooth and non degenerate.

Ideas of proof

- $\mathcal{H} \in C^{\alpha}(z)$ implies
  1. $z \mapsto \phi_a(z)$ $C^{1,\gamma}_{loc}$
  2. $a \mapsto \partial_z \phi_a(z)$ continous
  3. $|\partial_z \phi_a(z)| \geq C(|z|)\,|a| \Rightarrow$ Non Degeneracy (I)
- $\mathcal{H} \in C^{\alpha}(z) \cap C^{1,\beta}(w)$ implies
  1. $z \mapsto D_a\phi_a(z)$ $C^{1,\beta\gamma}_{loc}$
  2. $a \mapsto D_a\phi_a$ metrically continuous
  3. $D_z(D_a\phi_a) = D_a(D_z\phi_a)$ everywhere
  4. $a \mapsto D_z\phi_a$ metrically C^1
  5. $\det(D_a(\partial_z\phi_a)) \neq 0 \Rightarrow$ Non Degeneracy (II)

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Summarizing,

$$\begin{array}{rcl} \mathcal{H} \text{ u.p.} & \rightsquigarrow & \mathcal{F}_{\mathcal{H}} \\ \mathcal{H} \text{ u.p., smooth} & \rightsquigarrow & \mathcal{F}_{\mathcal{H}} \text{ smooth, non degenerate.} \end{array}$$

Theorem. If $\mathcal{F}$ is smooth and non degenerate then there is a unique $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$ such that every member of $\mathcal{F}$ solves $\mathcal{H}_{\mathcal{F}}$.

Corollary. If $\mathcal{H}$ is smooth and has u.p. then there is an involution

$$\mathcal{H} \quad \rightsquigarrow \quad \mathcal{F}_{\mathcal{H}} \quad \rightsquigarrow \quad \mathcal{H}_{\mathcal{F}_{\mathcal{H}}} = \mathcal{H}.$$

Open problems

- does $\mathcal{H}_{\mathcal{F}}$ has uniqueness?
- if so: is it true that $\mathcal{F}_{\mathcal{H}_{\mathcal{F}}} = \mathcal{F}$?
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Many thanks!