Characterizations of sets of finite perimeter: old and recent results

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Introduction

The study of sets of finite perimeter, going back to R.Caccioppoli in the '30 and to E.De Giorgi and H.Federer in the '50, marks the beginning of modern Geometric Measure Theory.

Sets of finite perimeter play an important role in the theory of minimal surfaces, capillarity problems, phase transitions, optimal partitions, etc. It is a class of sets sufficiently large to have compactness/completeness properties w.r.t. the $L^1_{\rm loc}$ convergence (i.e. local convergence in measure of the characteristic functions χ_E), sufficiently small to have good structural properties (density, isoperimetric properties, rectifiability of the measure-theoretic boundary, etc.).

In my talk I will review old and new characterizations of this class of sets, covering in particular two recent developments:

• a characterization of this class of sets in metric measure spaces $(X, d, \boldsymbol{m})$ (joint work with S.Di Marino).

• a new and somehow unexpected characterization of perimeter and sets of finite perimeter in Euclidean spaces (joint work with H.Brezis, J.Bourgain, A.Figalli).

Plan

1. Sets of finite perimeter and *BV* functions in Euclidean spaces
2. Non-distributional characterizations
3. *BMO*-type seminorms and sets of finite perimeter

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Sets of finite perimeter

We begin with a definition in the spirit of the theory of distributions, namely requiring the existence of the distributional derivative $D\chi_E$ of the characteristic function $\chi_E : \mathbb{R}^n \to \{0, 1\}$ of E.

Definition. (Caccioppoli, De Giorgi) Let $E \subset \mathbb{R}^n$ be a Borel set. We say that E has finite perimeter in $\mathbb{R}^n$ if there exists a vector-valued measure

$$D\chi_E = \big(D_1\chi_E, \ldots, D_n\chi_E\big)$$

with finite total variation in $\mathbb{R}^n$, satisfying

$$\int_E \frac{\partial \phi}{\partial x_i}\, dx = -\int \phi\, dD_i\chi_E \qquad \forall \phi \in C^1_c(\mathbb{R}^n),\ i = 1, \ldots, n.$$

An analogous definition can be given for BV functions, not necessarily characteristic functions, but we will mostly deal with sets. We will also use the traditional notation

$$P(E) = |D\chi_E|(\mathbb{R}^n), \qquad P(E, A) = |D\chi_E|(A) \quad A \in \mathscr{B}(\mathbb{R}^n).$$

Definitions by approximation

In the spirit of the $H = W$ identity for Sobolev spaces, we can also characterize sets of finite perimeter by approximation:

$$\exists\, E_h \text{ smooth}, E_h \to E \text{ locally in measure}, L := \limsup_{h\to\infty} \mathscr{H}^{n-1}(\partial E_h) < \infty.$$

This is close to Caccioppoli's original definition, based on polyhedral approximation (and then the optimal L is $P(E)$).

Alternatively, one can approximate by smooth functions:

$$\exists f_h \in C_c^\infty(\mathbb{R}^n), f_h \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n), L := \limsup_{h\to\infty} \int_{\mathbb{R}^n} |\nabla f_h|\, dx < \infty$$

(and then the optimal L is $P(E)$).

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Sets of finite perimeter in metric measure spaces

On a metric measure space (X, d, m) there is no natural notion of smooth set, but one can use Lipschitz functions as replacements of smooth functions.

Definition. $E \subset X$ Borel has finite perimeter if there exist $f_h : X \to \mathbb{R}$ locally Lipschitz, satisfying

$$\lim_{h\to\infty} f_h = \chi_E \quad \text{in } L^1(X) \qquad \text{and} \qquad \limsup_{h\to\infty} \int_X |\nabla f_h| \, d\boldsymbol{m} < \infty$$

(here $|\nabla f|$ is the so-called local Lipschitz constant).

One can then localize this construction, defining

$$P(E, A) = \inf\left\{ \liminf_{h\to\infty} \int_A |\nabla f_h| \, d\boldsymbol{m} : f_h \in \mathrm{Lip}_{\mathrm{loc}}(A), \ \lim_{h\to\infty} f_h = \chi_E \quad \text{in } L^1(A) \right\}$$

for $A \subset X$ open.

Theorem. (Miranda, A-Di Marino). *The set function $A \mapsto P(E, A)$ is the restriction to open sets of a Borel σ-additive measure.*

Structure theorem in $\mathbb{R}$ and in $\mathbb{R}^n$

In $\mathbb{R}$ the class of sets of finite perimeter can be characterized with elementary tools. Any of these sets is $\mathscr{L}^1$-equivalent to a finite union of connected subsets of $\mathbb{R}$, and

$$P(E) = \{\# \text{ of endpoints in } \mathbb{R}\}.$$

In order to get something similar in n dimensions, we can define the essential boundary $\partial^* E$ as the set of points where the density is neither 0 nor 1:

$$\partial^* E := \left\{ x \in \mathbb{R}^n : \limsup_{r \downarrow 0} \min\left\{ \frac{\mathscr{L}^n(B_r(x) \cap E)}{\mathscr{L}^n(B_r(x))}, \frac{\mathscr{L}^n(B_r(x) \setminus E)}{\mathscr{L}^n(B_r(x))} \right\} > 0 \right\}.$$

Structure theorem. (De Giorgi-Federer) *For any set E of finite perimeter the essential boundary $\partial^* E$ has finite $\mathscr{H}^{n-1}$-measure, is countably $\mathscr{H}^{n-1}$-rectifiable and*

$$P(E) = \mathscr{H}^{n-1}(\partial^* E).$$

More precisely, $P(E, \cdot) = \mathscr{H}^{n-1} \llcorner \partial^* E$, namely

$$P(E, A) = \mathscr{H}^{n-1}(A \cap \partial^* E) \qquad \forall A \in \mathscr{B}(\mathbb{R}^n).$$

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Behaviour along coordinate lines

A point of view pioneered by Beppo Levi, in connection with the Sobolev theory, is to look at the behaviour along all coordinate lines. Let $h \in \mathbb{R}^n$ be a coordinate vector and let us look at the 1-dimensional sections of E along lines parallel to h:

$$E_y := \{t \in \mathbb{R} : \ y + th \in E\} \qquad y \in h^\perp.$$

Then, the following result holds:

Theorem. *If, for all coordinate vectors h, E_y has finite perimeter for $\mathscr{H}^{n-1}$-a.e. $y \in h^\perp$ and*

$$\int_{h^\perp} P(E_y)\, d\mathscr{H}^{n-1}(y) < \infty$$

then E has finiter perimeter, and conversely.

A similar statement holds for BV functions and the proof is not too hard, using Fubini's theorem and a smoothing argument.

Federer's characterization

On the other hand, the next criterion is specific of sets and its proof is *much* more tricky:

Theorem. (Federer) *E has finite perimeter whenever $\mathscr{H}^{n-1}(\partial^* E) < \infty$.*

The proof is elementary in dimension $n = 1$, indeed the continuous function $f(t) = \int_0^t \chi_E(s)\,ds$ is differentiable and with derivative in $\{0,1\}$ out of finitely many points (those in $\partial^* E$), and one can apply the Darboux (mean value) property of derivative to obtain that the derivative is piecewise constant.

If, for h coordinate vector, we consider again the sections

$$E_y := \left\{t \in \mathbb{R} : \; y + th \in E\right\},$$

by the coarea inequality

$$\int_{h^\perp} \mathscr{H}^0\big((\partial^* E)_y\big)\, d\mathscr{H}^{n-1} \leq \mathscr{H}^{n-1}(\partial^* E)$$

we get $(\partial^* E)_y$ consists of finitely many points for $\mathscr{H}^{n-1}$-a.e. y.

Federer's characterization

The hard part of the proof is to show that, to some extent, $\partial^* E_y \subset (\partial^* E)_y$ for $\mathscr{H}^{n-1}$-a.e. y.

Although a posteriori all definitions are equivalent, the difficulty in Federer's criterion is due to the discrete-continuum-discrete-continuum procedure. Indeed, first to compute $\partial^* E$ we need to pass to an infinitesimal scale, so perform a discrete-to-continuum approximation. Then, to compute $\mathscr{H}^{n-1}(\partial^* E)$ we have to to the same thing.

In the metric measure setup, Federer's result has been extended by R.Korte, P.Lahti and N.Shanmugalingam, assuming doubling, $(1,1)$-Poincaré and the existence of "nice" fibrations of the metric measure space, adapting a concept introduced by Semmes.

A general criterion in metric measure spaces

Another characterization of sets of finite perimeter (and BV functions) can be obtained in general metric measure spaces $(X, d, \boldsymbol{m})$ without structural assumptions, by looking instead of $\partial^* E$ (which a priori might be even “large”, under no doubling assumptions), at the sections

$$E_\gamma := \left\{ t \in [0,1] : \ \gamma(t) \in E \right\}$$

of E along random curves $\gamma \in \mathrm{Lip}([0,1]; X)$.

Call $\boldsymbol{\pi} \in \mathscr{P}(C([0,1]; X))$ a test plan if:

(a) $\boldsymbol{\pi}$ is concentrated on $\mathrm{Lip}([0,1]; X)$ and $\mathrm{Lip}(\gamma) \in L^\infty\big(C([0,1]; X); \boldsymbol{\pi}\big)$;

(b) the marginals $(e_t)_\sharp \boldsymbol{\pi}$ satisfy $(e_t)_\sharp \boldsymbol{\pi} \ll \boldsymbol{m}$ and have densities uniformly bounded w.r.t. $\boldsymbol{m}$.

We denote by $C(\boldsymbol{\pi})$ the smallest constant satisfying $(e_t)_\sharp \boldsymbol{\pi} \leq C(\boldsymbol{\pi}) \boldsymbol{m}$ for all $t \in [0,1]$.

For instance, in $\mathbb{R}^n$, the collection of all “Levi" lines in a given direction, with variable parameterization, provide the support of a test plan.

A general criterion in metric measure spaces

Theorem. (A.-Di Marino) *$E \subset X$ has finite perimeter if and only if there exists a finite Borel measure μ in X satisfying*

$$(*) \qquad \int \mathscr{H}^0 \llcorner \{\gamma(t) :\ t \in \partial^* E_\gamma\}\, d\boldsymbol{\pi}(\gamma) \leq C(\boldsymbol{\pi}) \|\mathrm{Lip}(\gamma)\|_{L^\infty(\boldsymbol{\pi})}\, \mu$$

for all $\boldsymbol{\pi}$ test plan.
The smallest measure μ with this property is precisely $P(E,\cdot)$.

The proof that $P(E,\cdot)$ satisfies (*) is again a not too difficult Fubini-type argument. The proof of the converse implication is much more involved and uses ideas introduced in A-Gigli-Savaré in the context of optimal transportation and gradient flows.

A BMO-like characterization of the perimeter

In a recent paper, Bourgain, Brezis and Mironescu raised the question of finding the “largest” space X of functions $f : Q \to \mathbb{R}$ (and the weakest seminorm) satisfying the constancy theorem, namely

$$f \in X \text{ and } \mathbb{Z}\text{-valued} \qquad \Longrightarrow \qquad f = k \ \mathscr{L}^n\text{-a.e. in } Q, \text{ for some } k \in \mathbb{Z}.$$

It is easily seen that the spaces $VMO(Q)$, $W^{1,1}(Q)$ and the fractional Sobolev space $W^{1/p,p}(Q)$ with $1 < p < \infty$ all have this property, but no inclusion between these spaces exists. To this aim, they introduced the seminorm

$$[f] := \limsup_{\epsilon \downarrow 0} [f]_\epsilon,$$

where $[f]_\epsilon$ is defined with a suitable maximization procedure, and the space $B_0 = \{f : [f] = 0\}$. Then, they proved that all the spaces above are contained in B_0, and that the constancy theorem holds in B_0.

A BMO-like characterization of the perimeter

Set

$$[f]_\epsilon := \epsilon^{n-1} \sup_{\mathcal{G}_\epsilon} \sum_{Q \in \mathcal{F}_\epsilon} \fint_Q \left| f(x) - \fint_Q f \right| dx,$$

where the supremum runs among all families $\mathcal{G}_\epsilon$ of disjoint closed ϵ-cubes with faces parallel to the coordinate axes, with $\#\mathcal{G}_\epsilon \leq \epsilon^{1-n}$.

It was proved in BBM that $[\chi_E]$ is a good right hand side for the relative isoperimetric inequality:

$$\left(\fint_{Q_1} \left| \chi_E(x) - \fint_{Q_1} \chi_E \right|^{n/(n-1)} dx \right)^{(n-1)/n} \leq C(n)[\chi_{E \cap Q_1}].$$

This strongly suggests that $[\chi_E]$ and $P(E)$ should be closely related, although we should take into account that the former quantity $[f]$ is always finite when $f \in L^\infty$.

A BMO-like characterization of the perimeter

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\mathsf{I}_\epsilon(f) := \epsilon^{n-1} \sup_{\mathcal{F}_\epsilon} \sum_{Q \in \mathcal{F}_\epsilon} \fint_Q \left| f(x) - \fint_Q f \right| dx,$$

where the supremum runs among all families $\mathcal{F}_\epsilon$ of disjoint closed ϵ-cubes, not necessarily with faces parallel to the coordinate axes, with $\#\mathcal{F}_\epsilon \leq \epsilon^{1-n}$ (this is the isotropic variant of the quantities $[f]$, $[f]_\epsilon$ of BBM).

Theorem. (A-Brezis-Bourgain-Figalli) *For any Borel set $E \subset \mathbb{R}^n$, one has*

$$\lim_{\epsilon \downarrow 0} \mathsf{I}_\epsilon(\chi_E) = \frac{1}{2} \min\{1, P(E)\}$$

with the convention $P(E) = \infty$ if E has not finite perimeter.

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A BMO-like characterization of the perimeter

Since

$$\fint_Q \left| \chi_E(x) - \fint_Q \chi_E \right| dx = 2\frac{\mathscr{L}^n(E\cap Q)}{\mathscr{L}^n(Q)}\left(1 - \frac{\mathscr{L}^n(E\cap Q)}{\mathscr{L}^n(Q)}\right) \le \frac{1}{2},$$

we get $\mathsf{I}_\epsilon(\chi_E) \le 1/2$. Hence, the result should be interpreted as a treshold phenomenon: as soon as $\mathsf{I}_\epsilon(\chi_E)$ goes below the obvious treshold, from

$$\lim_{\epsilon\downarrow 0} \mathsf{I}_\epsilon(\chi_E) = \frac{1}{2}\min\{1, P(E)\}$$

we gain regularity, more precisely that the perimeter of E is finite, and that $P(E) \le 1$.

The validity of the result, and the presence of the factor $1/2$ are related, as we will see, to the sharp constant for the relative isoperimetric inequality in the cube.

Variants

(1) If we define $\mathsf{J}_\epsilon \geq \mathsf{I}_\epsilon$ by maximizing on finite, disjoint families $\mathcal{F}_\epsilon$ of closed and tilted ϵ-cubes, but with no constraint on their cardinality, we get from the previous theorem

$$\lim_{\epsilon \downarrow 0} \mathsf{J}_\epsilon(\chi_E) = \frac{P(E)}{2}.$$

Howewer, this result is weaker and much easier to prove, since it hides the "treshold" phenomenon.

(2) Considering the original anisotropic quantities $[f]_\epsilon$ of BBM, we expect a similar result, but with an isotropic notion of perimeter, a concept already well estabilished in the literature.

(3) Using the sharp constant for the relative isoperimetric inequality in balls and families of disjoint ϵ-balls, we expect a similar result to be true (but this makes the link with BMO and VMO weaker).

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Sketch of proof: the inequality $\leq$

We already remarked that $\mathsf{I}_\epsilon(\chi_E) \leq 1/2$. Hence, $2\limsup_\epsilon \mathsf{I}_\epsilon(\chi_E) \leq \min\{1, P(E)\}$ follows by

$$(*) \qquad 2\limsup_{\epsilon\downarrow 0} \mathsf{I}_\epsilon(\chi_E) \leq P(E).$$

In order to obtain (*) we can assume $P(E) < \infty$ and just notice that the relative isoperimetric inequality in the cube (Hadwiger) gives

$$\mathscr{L}^n(A)(1-\mathscr{L}^n(A)) \leq \frac{1}{4}P(A, \mathring{Q}_1) \quad \text{for any } A \text{ contained in the 1-cube } Q_1,$$

so that by scaling we get

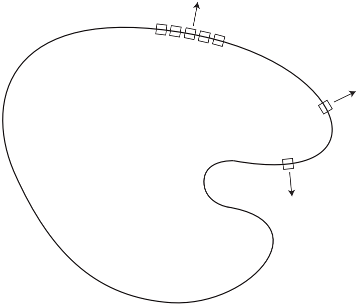
$$\epsilon^{n-1}\int_Q \left|\chi_E(x) - \fint_Q \chi_E\right| dx = 2\epsilon^{n-1}\frac{\mathscr{L}^n(E\cap Q)}{\mathscr{L}^n(Q)}\left(1-\frac{\mathscr{L}^n(E\cap Q)}{\mathscr{L}^n(Q)}\right) \leq \frac{1}{2}P(E, \mathring{Q})$$

for any ϵ-cube Q.

We now need only to add on all cubes Q of the family $\mathcal{F}_\epsilon$ to obtain (*).

Sketch of proof: the inequality $\geq$ when $P(E) < \infty$

When $P(E) < \infty$ we know that $\partial^* E$ is countably $\mathscr{H}^{n-1}$-rectifiable, so on small scales, near to $\mathscr{H}^{n-1}$-a.e. point, $\partial^* E$ is close to an halfspace and E is close to an halfplane. In this case, it is heuristically clear that the optimal choice of the cubes is as in the picture:



However, we have to be careful, because we have a *global* bound ϵ^{1-n} on the number of cubes that we have to choose when working on an ϵ-scale.

Sketch of proof: the inequality $\geq$ when $P(E) < \infty$

The correct choice of the number of cubes is somehow dictated by the perimeter itself, so we work with the localized quantities

$$\mathsf{J}_\epsilon(E,\Omega) = \epsilon^{n-1} \sup_{\mathcal{F}_\epsilon} \sum_{Q \in \mathcal{F}_\epsilon} \fint_Q \left| \chi_E(x) - \fint_Q \chi_E \right| dx$$

where, this time, we consider only cubes Q contained in Ω and the cardinality of $\mathcal{F}_\epsilon$ does not exceed the scale-invariant quantity $P(E,\Omega)\epsilon^{1-n}$. Then, a blow-up analysis provides the result.

Sketch of proof: the inequality $\geq$ when $P(E) = \infty$

This is one of the most delicate parts of the proof. We have to show that

$$(*) \qquad P(E) = \infty \quad \Longrightarrow \quad \liminf_{\epsilon \downarrow 0} \mathsf{I}_\epsilon(\chi_E) \geq \frac{1}{2}.$$

I will describe a soft argument which proves the existence of constants $\xi(n)$, $\eta(n)$ satisfying

$$(**) \qquad \limsup_{\epsilon \downarrow 0} \mathsf{I}_\epsilon(\chi_E) < \xi(n) \quad \Longrightarrow \quad P(E) \leq \eta(n) \limsup_{\epsilon \downarrow 0} \mathsf{I}_\epsilon(\chi_E)$$

which gives (*) with $\xi(n)$ in place of $1/2$.

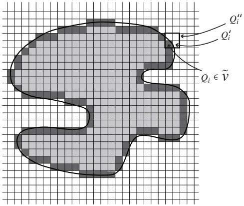
This already illustrates the "treshold" phenomenon, but with non-optimal constants, and the actual proof is a (nontrivial) refinement of this idea.

Sketch of proof: the inequality $\geq$ when $P(E) = \infty$

Let us consider the canonical dyadic subdivision (up to a Lebesgue negligible set) of $(0,1)^n$ in 2^{hn} cubes Q_i with length side 2^{-h}. We define on the scale $\epsilon = 2^{-h}$ an approximate interior $\mathrm{Int}_h(E)$ of E by considering the set

$$I_h := \left\{ i \in \{1, \ldots, 2^{hn}\} : \ \fint_{Q_i} \chi_E > \frac{3}{4} \right\}$$

and taking the union of the cubes Q_i, $i \in I_h$. Analogously we define a set of indices E_h and the corresponding approximate exterior $\mathrm{Ext}_h(E) = \mathrm{Int}_h(Q \setminus E)$. We denote by F_h the complement of $I_h \cup E_h$ and by $\mathrm{Bdry}_h(Q)$ the union of the corresponding cubes.



Sketch of proof: the inequality $\geq$ when $P(E) = \infty$

Since $\mathrm{Int}_h(E) \to E$ in L^1_{loc} as $h \to \infty$, by the lower semicontinuity of the perimeter it suffices to give a uniform estimate on $P(\mathrm{Int}_h(E)) = \mathscr{H}^{n-1}(\partial\,\mathrm{Int}_h(E))$ as $h \to \infty$ under a smallness assumption on $\limsup_h \mathsf{I}_{2^{-h}}(\chi_E)$. Since $\fint_{Q_i} |\chi_E(x) - \fint_{Q_i} \chi_E|\,dx \geq 1/4$ for all $i \in F_h$ (by definition of F_h), we obtain that

$$\mathsf{I}_{2^{-h}}(\chi_E) < \frac{1}{4} \qquad \Longrightarrow \qquad \#\,F_h \leq 4\,\mathsf{I}_{2^{-h}}(\chi_E)\,(2^{-h})^{1-n} < (2^{-h})^{1-n},$$

which provides a uniform estimate on $\mathscr{H}^{n-1}(\partial\,\mathrm{Bdry}_h(E))$. Hence, to control $\mathscr{H}^{n-1}(\partial\,\mathrm{Int}_h(E))$ it suffices to bound the number of faces $F \subset Q$ common to a cube Q_i and a cube Q_j, with $i \in I_h$ and $j \in E_h$. For this, notice that if Q is the parent cube with side length 2^{1-h} containing $Q_i \cup Q_j$, it is easily seen that

$$\fint_Q \left|\chi_E(x) - \fint_Q \chi_E\right| dx \geq 2^{-1-n}$$

and this leads once more to an estimate of the number of these cubes with $(2^{1-h})^{1-n}$ provided $\mathsf{I}_{2^{1-h}}(\chi_E) < 2^{-1-n}$. Combining this estimate with the uniform estimate on $\mathscr{H}^{n-1}(\partial\,\mathrm{Bdry}_h(E))$ leads to (**).

Essential bibliography

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